1. (6 pts) Give the definitions of the following notions:
   a) Asymptotically stable equilibrium of a system of differential equations;
   b) A homogeneous linear differential equation;

Solution.

a) $y \equiv y_0$ is an asymptotically stable equilibrium of $y' = f(y)$ if it's a stable equilibrium and in addition there exists $\epsilon > 0$ such that any solution of $y' = f(y)$ with $|y(0) - y_0| < \epsilon$ satisfies $y(t) \to y_0$ as $t \to +\infty$.

b) A homogeneous linear differential equation is an ODE of the form

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_0(t)y = 0$$

2. (10 pts) Mark true or false. If true, give an argument why, if false, give a counterexample.
   a) If A, B are $n \times n$ real matrices such that $e^A = e^B$ then $A = B$.
   b) If A is upper triangular then e^A is also upper triangular.

Solution.

a) **False.**
For example, if $A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & \beta + 2\pi k \\ -\beta - 2\pi k & 0 \end{pmatrix}$ then $e^A = e^B = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}$ for any integer k.

b) **True.**
For any upper triangular matrices A, B their product AB and sum $A + B$ are also upper triangular. Therefore A^n is upper triangular for any n. Hence, $I + A + \ldots + \frac{A^n}{n!}$ is upper triangular for any n and so is the limit $\lim_{n\to\infty}(I + A + \ldots + \frac{A^n}{n!}) = e^A$.

3. (8 pts) Using the variation of parameter find the general solution of the following equation:

$$y'' - 2y' + y = e^t$$

Note: Solutions using methods other than the variation of parameter will be awarded zero credit!

Solution.

First, we solve the homogeneous equation $y'' - 2y' + y = 0$. The corresponding characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$ which has a double root $\lambda = 1$. Therefore, the general solution of $y'' - 2y' + y = 0$ is $y = c_1e^t + c_2te^t$. The Wronskian of the solutions $y_1 = e^t, y_2 = te^t$

is equal to

$$W = \det \begin{pmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{pmatrix} = e^{2t}$$

By the variation of parameter method we look for the solution of $y'' - 2y' + y = e^t$ in the form $y = c_1(t)e^t + c_2(t)te^t$. Here $c_1(t) = \int -\frac{e^t te^t}{W} = \int -tdt = -\frac{t^2}{2} + \tilde{c}_1$. Similarly, $c_2(t) = \int \frac{e^t e^t}{W} = \int dt = t + \tilde{c}_2$. Therefore, the general solution of $y'' - 2y' + y = e^t$ is given by

$$y = (-\frac{t^2}{2} + \tilde{c}_1)e^t + (t + \tilde{c}_2)te^t = \boxed{\frac{t^2}{2} + \tilde{c}_1 e^t + \tilde{c}_2 te^t}.$$

4. (8 pts) Consider a linear system of differential equations $y' = A(t)y$ where $A(t)$ is a C^∞ family of real $n \times n$ matrices.

   Prove that the solution space of this system is a real vector space of dimension n.

   You can use without a proof the fact that any solution of such a system exists for all $t \in R$.

Solution.

Let W be the solution space. Then W is a subset of the vector space of vector-valued functions therefore it's enough to check that it's a vector subspace. To this end we need to check that it's closed under addition and multiplication by scalars.

If $y_1(t), y_2(t) \in W$ then $y_1' = A(t)y_1$ and $y_2' = A(t)y_2$. Adding these equations we get that $(y_1 + y_2)' = A(t)(y_1 + y_2)$ which means that $y_1 + y_2$ also lies in W.

Similarly, if $y \in W$ then λy is also in W for any constant λ. Thus W is a vector space.

To compute its dimension we'll construct an isomorphism from W to $\mathbb{R}^n$.

Consider the following map $L\colon W \to \mathbb{R}^n$: $L(y) := y(0)$. This map is obviously linear. It's 1-1 by the uniqueness theorem and it's onto by the existence theorem for linear IVP. Therefore, L is an isomorphism and hence $\dim W = \dim \mathbb{R}^n = n$.

5. (10 pts) Find all equilibrium points of the following system of differential equations and describe the type of the linearized system for each of them.

$$\begin{cases} x' = x - y^2 \\ y' = x + 2y - 3 \end{cases}$$

Solution.

To find the equilibrium points we solve

$$\begin{cases} x - y^2 = 0 \\ x + 2y - 3 = 0 \end{cases} \qquad \begin{cases} x = y^2 \\ y^2 + 2y - 3 = 0 \end{cases} \qquad \begin{cases} x = y^2 \\ y = -3, 1 \end{cases}$$

Thus, the equilibrium points are $(9, -3)$ and $(1, 1)$.

The linearized system at any point is given by $y' = Ay$ where A is the matrix of partial derivatives at this point. We first compute the matrix of partial derivatives at any point

$$\frac{\partial f_i}{\partial x_j} = \begin{pmatrix} 1 & -2y \\ 1 & 2 \end{pmatrix}$$

therefore at the point $(9, -3)$ the linearized system is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 6 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Computing the eigenvalues of A at this point we find that $\lambda_1 = -1, \lambda_2 = 4$ which means that the linearized system is a $\boxed{\text{saddle}}$.

Similarly, computing A at (1,1) we find $A = \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}$. this gives $\lambda = \frac{3 \pm \sqrt{7} i}{2}$ which means that the linearized system is a $\boxed{\text{spiral source}}$.

6. (10 pts) Consider the following system of ODEs.

$$\begin{cases} x' = (-5z + 2x)(y - 1) \\ y' = -y^3(x^2 + 1) \\ z' = (z + 2x)(y - 1) \end{cases} \tag{1}$$

Find its equilibrium points and determine whether or not they are stable.

Solution.

Solving

$$\begin{cases} 0 = (-5z + 2x)(y - 1) \\ 0 = -y^3(x^2 + 1) \\ 0 = (z + 2x)(y - 1) \end{cases} \tag{2}$$

We find that $(0, 0, 0)$ is the only solution and hence is the only equilibrium point. The linearized system at the origin has the matrix

$$A = \begin{pmatrix} -2 & 0 & 5 \\ 0 & 0 & 0 \\ -2 & 0 & -1 \end{pmatrix}$$

which is not hyperbolic. Hence, to determine stability, we need to look for a Liapunov function. We look for L in the form $L = ax^2 + by^2 + cz^2$. If $x(t), y(t), z(t)$ is a solution then

$L(x,y,z)' = 2axx' + 2byy' + 2czz' = 2ax(-5z + 2x)(y-1) - 2by^4(x^2+1) + 2cz(z+2x)(y-1) = 2(y-1)(2ax^2 + cz^2 + (2c - 5a)xz) - 2by^4(x^2+1)$.

Choosing $a = 5, c = 2, b = 1$ this gives $L(x,y,z)' = 2(y-1)(5x^2 + 2z^2) - 2y^4(x^2+1) \leq 0$ if x, y, z is close to the origin (since $y - 1 < 0$ for y near 0). Moreover, $L(x,y,z)' < 0$ near $(0,0,0)$ unless $(x,y,z) = (0,0,0)$.

Therefore, $(0,0,0)$ is an $\boxed{\text{asymptotically stable equilibrium point}}$.

7. (8 pts) Let v be a generalized eigenvector of an $n \times n$ matrix A. Prove that $e^A \cdot v$ is also a generalized eigenvector for A.

   Hint: Use that $e^A B = B e^A$ if $AB = BA$.

Solution.

By definition of a generalized eigenvector, there exists $k > 0$ such that $(A - \lambda I)^k v = 0$ where λ is some eigenvalue of A. Clearly, $(A - \lambda I)A = A(A - \lambda I)$. Hence, $(A - \lambda I)^k A = A(A - \lambda I)^k$ and therefore $(A - \lambda I)^k e^A = e^A (A - \lambda I)^k$.

Applying this to v we get $(A - \lambda I)^k e^A v = e^A (A - \lambda I)^k v = 0$ which means that $e^A v$ is a generalized eigenvector of A. $\square$

8. (8 pts) Find the general solution of the following differential equation

$$(y^2 e^{xy^2} + y^3) + (2xye^{xy^2} + 3xy^2 + 2y)\frac{dy}{dx} = 0$$

Solution.

Let $M = (y^2 e^{xy^2} + y^3)$ and $N = (2xye^{xy^2} + 3xy^2 + 2y)$. A direct calculation shows that $M_y = N_x$ which means that the above ODE is exact. This means that there exists $F(x,y)$ such that $M = F_x$ and $N = F_y$. To find F we integrate $F = \int M dx + C(y) = \int (y^2 e^{xy^2} + y^3) dx + C(y) = e^{xy^2} + xy^3 + C(y)$.

To find $C(y)$ we differentiate in y and get $F_y = e^{xy^2} 2xy + 3xy^2 + C'(y) = N = 2xye^{xy^2} + 3xy^2 + 2y$. This gives $C'(y) = 2y, C(y) = y^2 + const.$

Thus the general solution of the ODE is given by

$\boxed{e^{xy^2} + xy^3 + y^2 = \tilde{C} \text{ where } \tilde{C} \text{ is an arbitrary constant}}$.

9. (6 pts) Show the solution of the IVP

$$\begin{cases} x' = \sin(x+2y) \\ y' = e^{-x^2} \\ x(0) = x_0 \\ y(0) = y_0 \end{cases}$$

exists for all $t \geq 0$ for any initial condition (x_0, y_0).

Solution.

Two different solutions are possible.

Solution 1. Let (t_1, t_2) be the maximal interval of existence of the solution (here $t_1 < 0 < t_2$). Suppose t_2 is finite. By the theorem on extending solutions this can only happen if $\sup_{t_0 \leq t < t_2} |(x(t), y(t))| = +\infty$. However, $|x'| = |\sin(2x+y)| \leq 1$ for all t which means that $|x(t)| \leq |x_0| + |t|$ for any t in the interval of existence. Therefore, $|x(t)| \leq |x_0| + t_2$ for any $t \in [0, t_2)$. Similarly, $|y'| = |e^{-x^2}| \leq 1$ which means that $|y(t)| \leq |y_0| + t_2$ for any $t \in [0, t_2)$. This means that $\sup_{t_0 \leq t < t_2} |(x(t), y(t))| < \infty$ which is a contradiction. Hence $t_2 = +\infty$. The same argument shows that $t_1 = -\infty$. □

Sketch of Solution 2. Arguing as in the problem 3 from HW 9, we can show that the interval of existence of

$$\begin{cases} x' = \sin(x+2y) \\ y' = e^{-x^2} \\ x(t_0) = x_0 \\ y(t_0) = y_0 \end{cases}$$

is the same for all t_0, x_0, y_0 which immediately implies that any solution extends to the whole $\mathbb{R}$.

Fill in the details in the above argument.

10. (8 pts) Let A be a 2×2 real matrix of the form $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Consider the system $y' = Ay$.

Describe all possible phase portraits this system can have depending on a and b.

Solution.

The eigenvalues of A are $\lambda = a \pm ib$. Thus we get the following cases:

(a) $b \neq 0, a > 0$ - spiral source;
(b) $b \neq 0, a < 0$ - spiral sink;
(c) $b \neq 0, a = 0$- center;
(d) $b = 0, a > 0$- source;
(e) $b = 0, a < 0$- sink;

(f) $a = b = 0$. This is a degenerate case when all points in $\mathbb{R}^2$ are equilibrium points.

11. (10 pts) Consider the following IVP

$$\begin{cases} y' = y^2 + y \\ y(0) = 1 \end{cases}$$

(a) Write the first two terms of the Picard iteration process;

(b) Find an $a > 0$ such that the above system is guaranteed to have a unique solution for $0 \leq t \leq a$ satisfying $0 \leq y \leq 2$.

Solution.

(a) We have $y_n(t) = y(0) + \int_0^t f(y(x))dx$ and $y_0(t) \equiv y(0) = 1$. This gives $y_1(t) = 1 + \int_0^t (1^2+1)dx = 1+2t$. Similarly, $y_2(t) = 1 + \int_0^t (1+2x)^2 + 1 + 2xdx = 1 + \int_0^t (4x^2+6x+2)dx = 1 + 4t^3/3 + 3t^2 + 2t$.

(b) Let $R = 1$. We are looking for the existence interval where $|y(t)-1| < 1 = R$ (i.e $0 < y < 2$). Let $M = \max_{0\le y\le 2}|y^2+2y| = 2^2+2\cdot 2 = 8$. Let $K = \max_{0\le y\le 2}|f'(y)| = \max_{0\le y\le 2}|2y+1| = 5$. By the existence theorem, for any $0 < a < \min\{\frac{1}{K}, \frac{R}{M}\} = \min\{1/5, 1/8\} = \boxed{1/8}$ the solution exists for $-a < t < a$ and it satisfies $0 < y(t) < 2$.

12. (8 pts) Consider the following IVP

$$\begin{cases} y' = (\sin t)y + t \\ y(0) = 1 \end{cases}$$

Show that $y(t) \ge t - 1 + 2e^{-t}$ for $t \ge 0$. You can assume that $y(t) \ge 0$ for $t \ge 0$.

Extra credit (3 pts): Prove that $y(t) \ge t - 1 + 2e^{-t}$ for $t \ge 0$ without assuming that $y(t) \ge 0$ for $t \ge 0$.

Solution.

If $y \ge 0$ then $y' = (\sin t)y + t \ge -y + t$. Hence $y' + y \ge t$. We multiply this by e^t.

$$e^t y' + e^t y \ge te^t, \quad (e^t y)' \ge te^t, \quad \int_0^t (e^t y)' dt \ge \int_0^t te^t dt, \quad e^t y(t) - y(0) \ge (t-1)e^t + 1$$

$$e^t y(t) \ge (t-1)e^t + 2, \quad y(t) \ge t - 1 + 2e^{-t} \quad \square$$

To do the problem without assuming $y(t) \ge 0$. Suppose $y(t) \le 0$ for some $t > 0$. Let t_0 be the first point where $y(t) = 0$ so that $y(t) > 0$ on $[0, t_0)$ and $y(t_0) = 0$. Then the above argument shows that $y(t) \ge t - 1 + 2e^{-t}$ for $0 \le t \le t_0$. In particular, $y(t_0) \ge t_0 - 1 + 2e^{-t_0}$. On the other hand, it's easy to see (exercise in calculus) that $t - 1 + 2e^{-t} > 0$ for all $t \ge 0$ which means that $y(t_0) > 0$. This is a contradiction and therefore $y(t) > 0$ for all $t \ge 0$.