1. (6 pts) Give the definitions of the following notions:
   a) Asymptotically stable equilibrium of a system of differential equations;
   b) A homogeneous linear differential equation;
2. (10 pts) Mark true or false. If true, give an argument why, if false, give a counterexample.
   a) If A, B are $n \times n$ real matrices such that $e^A = e^B$ then $A = B$.
   b) If A is upper triangular then e^A is also upper triangular.
3. (8 pts) Using the variation of parameter find the general solution of the following equation:

$$y'' - 2y' + y = e^t$$

   Note: Solutions using methods other than the variation of parameter will be awarded zero credit!
4. (8 pts) Consider a linear system of differential equations $y' = A(t)y$ where $A(t)$ is a C^∞ family of real $n \times n$ matrices.

   Prove that the solution space of this system is a real vector space of dimension n.

   You can use without a proof the fact that any solution of such a system exists for all $t \in R$.
5. (10 pts) Find all equilibrium points of the following system of differential equations and describe the type of the linearized system for each of them.

$$\begin{cases} x' = x - y^2 \\ y' = x + 2y - 3 \end{cases}$$

6. (10 pts) Consider the following system of ODEs.

$$\begin{cases} x' = (-5z + 2x)(y - 1) \\ y' = -y^3(x^2 + 1) \\ z' = (z + 2x)(y - 1) \end{cases} \tag{1}$$

   Find its equilibrium points and determine whether or not they are stable.
7. (8 pts) Let v be a generalized eigenvector of an $n \times n$ matrix A. Prove that $e^A \cdot v$ is also a generalized eigenvector for A.

   Hint: Use that $e^A B = B e^A$ if $AB = BA$.
8. (8 pts) Find the general solution of the following differential equation

$$(y^2 e^{xy^2} + y^3) + (2xye^{xy^2} + 3xy^2 + 2y)\frac{dy}{dx} = 0$$

9. (6 pts) Show the solution of the IVP

$$\begin{cases} x' = \sin(x + 2y) \\ y' = e^{-x^2} \\ x(0) = x_0 \\ y(0) = y_0 \end{cases}$$

exists for all $t \geq 0$ for any initial condition (x_0, y_0).

10. (8 pts) Let A be a 2×2 real matrix of the form $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Consider the system $y' = Ay$.

    Describe all possible phase portraits this system can have depending on a and b.

11. (10 pts) Consider the following IVP

$$\begin{cases} y' = y^2 + y \\ y(0) = 1 \end{cases}$$

    (a) Write the first two terms of the Picard iteration process;
    (b) Find an $a > 0$ such that the above system is guaranteed to have a unique solution for $0 \leq t \leq a$ satisfying $0 \leq y \leq 2$.

12. (8 pts) Consider the following IVP

$$\begin{cases} y' = (\sin t)y + t \\ y(0) = 1 \end{cases}$$

    Show that $y(t) \geq t - 1 + 2e^{-t}$ for $t \geq 0$. You can assume that $y(t) \geq 0$ for $t \geq 0$.

    Extra credit (3 pts): Prove that $y(t) \geq t - 1 + 2e^{-t}$ for $t \geq 0$ without assuming that $y(t) \geq 0$ for $t \geq 0$.