(1) Let U be an orthogonal $n \times n$ matrix with $\det(U) = 1$. Prove that there exists a skew-symmetric real matrix A such that $U = e^A$.

Solution.

First observe that U is normal since it commutes with $U^\star = U^T$ as by definition of an orthogonal matrix $UU^\star = U^\star U = I$.

Therefore U admits an orthonormal basis of complex eigenvectors, $u_1, \ldots, u_n$. In other words, U can be written as $U = TDT^{-1}$ where D is diagonal and T is unitary (columns of T are given by $u_1, \ldots, u_n$).

Next observe that all eigenvalues of U have absolute value 1.

Indeed, if $Uv = \lambda v$ for some $v \neq 0$ then $\langle Uv, Uv \rangle = \langle \lambda v, \lambda v \rangle = |\lambda|^2 \cdot |v|^2$. On the other hand $\langle Uv, Uv \rangle = \langle v, U^\star U v \rangle = \langle v, v \rangle = |v|^2$ and hence $|\lambda| = 1$. In particular, if λ is real then $\lambda = \pm 1$.

Let's put U into real canonical form using T. Since for a normal matrix, eigenvectors corresponding to distinct eigenvalues are orthogonal we can assume that our basis $u_1, \ldots, u_n$ looks like $v_1, \bar{v}_1, \ldots, v_k, \bar{v}_k, v_{k+1}, \ldots v_m$. Where $v_1, \ldots v_k$ are complex eigenvectors corresponding to complex eigenvalues $\lambda_1, \ldots, \lambda_k$ and $v_{k+1}, \ldots v_m$ are real eigenvectors corresponding to real eigenvalues ± 1. Note that since $|\lambda_i| = 1$ we can write it as $\lambda_i = \cos \alpha_i + i \sin \alpha_i$.

Next observe that if $v = u + iw$ where both u and w are real and $v \perp \bar{v}$ then $u \perp w$ and $|u| = |w|$.

Indeed, $0 = \langle v, \bar{v} \rangle = \langle u + iw, u - iw \rangle = |u|^2 - |w|^2 + 2i \langle w, u \rangle$ so that $|u|^2 - |w|^2 = 0$ and $\langle w, u \rangle = 0$.

Therefore we can assume that the vectors $u_1, w_1, \ldots, u_k, w_k, v_{k+1}, \ldots, v_m$ are orthonormal.

Let Q be the matrix with columns $u_1, w_1, \ldots, u_k, w_k, v_{k+1}, \ldots, v_m$. Then Q is orthogonal and $U = QJ_RQ^{-1}$ where J_R is the real canonical form of U. By above J_R has block-diagonal form where the first k blocks are 2×2 matrices of the form

$$\begin{pmatrix} \cos \alpha_i & \sin \alpha_i \\ -\sin \alpha_i & \cos \alpha_i \end{pmatrix}$$

and the remaining blocks are 1×1 equal to ± 1. Note that the number of -1s is even since otherwise $\det U = \det J_R$ would be negative. We can collect -1's in pairs and write them in 2×2 blocks as $\begin{pmatrix} \cos \pi & \sin \pi \\ -\sin \pi & \cos \pi \end{pmatrix}$. Thus we can assume that the 1×1 blocks are all 1s.

Then $U = e^A$ where $A = QBQ^{-1}$, Q is orthogonal and B is block-diagonal with 2×2 blocks of the form $\begin{pmatrix} 0 & \alpha_i \\ -\alpha_i & 0 \end{pmatrix}$ and all 1×1 blocks equal to 0.

Note that B is skew-symmetric and hence so is A.

Indeed, $A^T = (QBQ^{-1})^T = (Q^{-1})^T B^T Q^T = Q(-B)Q^{-1} = -A$.