CHAPTER
2

First Order Differential Equations

This chapter deals with differential equations of first order,

$$\frac{dy}{dt} = f(t, y), \tag{1}$$

where f is a given function of two variables. Any differentiable function $y = \phi(t)$ that satisfies this equation for all t in some interval is called a solution, and our object is to determine whether such functions exist and, if so, to develop methods for finding them. Unfortunately, for an arbitrary function f, there is no general method for solving the equation in terms of elementary functions. Instead, we will describe several methods, each of which is applicable to a certain subclass of first order equations. The most important of these are linear equations (Section 2.1), separable equations (Section 2.2), and exact equations (Section 2.6). Other sections of this chapter describe some of the important applications of first order differential equations, introduce the idea of approximating a solution by numerical computation, and discuss some theoretical questions related to existence and uniqueness of solutions. The final section deals with first order difference equations, which have some important points of similarity with differential equations and are in some respects simpler to investigate.

2.1 Linear Equations with Variable Coefficients

If the function f in Eq. (1) depends linearly on the dependent variable y, then Eq. (1) is called a first order linear equation. In Sections 1.1 and 1.2 we discussed a restricted type of first order linear equation in which the coefficients are constants. A typical example is

$$\frac{dy}{dt} = -ay + b, \tag{2}$$

where a and b are given constants. Recall that an equation of this form describes the motion of an object falling in the atmosphere. Now we want to consider the most general first order linear equation, which is obtained by replacing the coefficients a and b in Eq. (2) by arbitrary functions of t. We will usually write the general **first order linear equation** in the form

$$\frac{dy}{dt} + p(t)y = g(t), \tag{3}$$

where p and g are given functions of the independent variable t.

Equation (2) can be solved by the straightforward integration method introduced in Section 1.2. That is, we rewrite the equation as

$$\frac{dy/dt}{y - (b/a)} = -a. \tag{4}$$

Then, by integration we obtain

$$\ln|y - (b/a)| = -at + C,$$

from which it follows that the general solution of Eq. (2) is

$$y = (b/a) + ce^{-at}, \tag{5}$$

where c is an arbitrary constant. For example, if $a = 2$ and $b = 3$, then Eq. (2) becomes

$$\frac{dy}{dt} + 2y = 3, \tag{6}$$

and its general solution is

$$y = \tfrac{3}{2} + ce^{-2t}. \tag{7}$$

Unfortunately, this direct method of solution cannot be used to solve the general equation (3), so we need to use a different method of solution for it. The method is due to Leibniz; it involves multiplying the differential equation (3) by a certain function $\mu(t)$, chosen so that the resulting equation is readily integrable. The function $\mu(t)$ is called an **integrating factor** and the main difficulty is to determine how to find it. To make the initial presentation as simple as possible, we will first use this method to solve Eq. (6), later showing how to extend it to other first order linear equations, including the general equation (3).

EXAMPLE 1

Solve Eq. (6),

$$\frac{dy}{dt} + 2y = 3,$$

by finding an integrating factor for this equation.

The first step is to multiply Eq. (6) by a function $\mu(t)$, as yet undetermined; thus

$$\mu(t)\frac{dy}{dt} + 2\mu(t)y = 3\mu(t). \tag{8}$$

The question now is whether we can choose $\mu(t)$ so that the left side of Eq. (8) is recognizable as the derivative of some particular expression. If so, then we can integrate Eq. (8), even though we do not know the function y. To guide our choice of the integrating factor $\mu(t)$, observe that the left side of Eq. (8) contains two terms and that the first term is part of the result of differentiating the product $\mu(t)y$. Thus, let us try to determine $\mu(t)$ so that the left side of Eq. (8) becomes the derivative of the expression $\mu(t)y$. If we compare the left side of Eq. (8) with the differentiation formula

$$\frac{d}{dt}[\mu(t)y] = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y, \tag{9}$$

we note that the first terms are identical and that the second terms also agree, provided we choose $\mu(t)$ to satisfy

$$\frac{d\mu(t)}{dt} = 2\mu(t). \tag{10}$$

Therefore our search for an integrating factor will be successful if we can find a solution of Eq. (10). Perhaps you can readily identify a function that satisfies Eq. (10): What function has a derivative that is equal to two times the original function? More systematically, rewrite Eq. (10) as

$$\frac{d\mu(t)/dt}{\mu(t)} = 2, \tag{11}$$

which is equivalent to

$$\frac{d}{dt}\ln|\mu(t)| = 2. \tag{12}$$

Then it follows that

$$\ln|\mu(t)| = 2t + C, \tag{13}$$

or

$$\mu(t) = ce^{2t}. \tag{14}$$

The function $\mu(t)$ given by Eq. (14) is the integrating factor for Eq. (6). Since we do not need the most general integrating factor, we will choose c to be one in Eq. (14) and use $\mu(t) = e^{2t}$.

Now we return to Eq. (6), multiply it by the integrating factor e^{2t}, and obtain

$$e^{2t}\frac{dy}{dt} + 2e^{2t}y = 3e^{2t}. \tag{15}$$

By the choice we have made of the integrating factor, the left side of Eq. (15) is the derivative of $e^{2t}y$, so that Eq. (15) becomes

$$\frac{d}{dt}(e^{2t}y) = 3e^{2t}. \tag{16}$$

By integrating both sides of Eq. (16) we obtain

$$e^{2t}y = \tfrac{3}{2}e^{2t} + c \tag{17}$$

where c is an arbitrary constant. Finally, on solving Eq. (17) for y, we have the general solution of Eq. (6), namely,

$$y = \tfrac{3}{2} + ce^{-2t}. \tag{18}$$

Of course, the solution (18) is the same as the solution (7) found earlier. Figure 2.1.1 shows the graph of Eq. (18) for several values of c. The solutions converge to the equilibrium solution $y = 3/2$, which corresponds to $c = 0$.

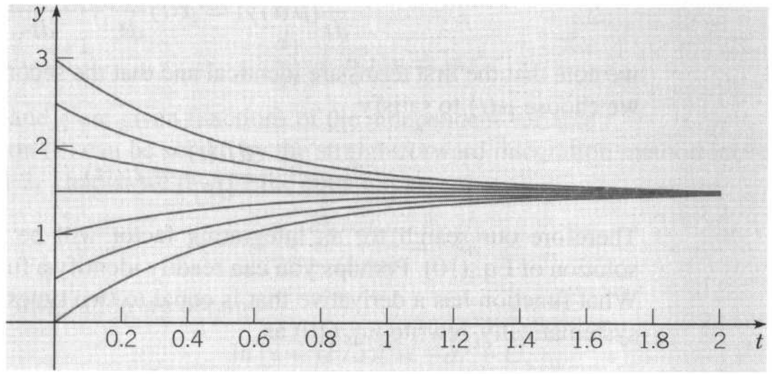

FIGURE 2.1.1 Integral curves of $y' + 2y = 3$.

Now that we have shown how the method of integrating factors works in this simple example, let us extend it to other classes of equations. We will do this in three stages. First, consider again Eq. (2), which we now write in the form

$$\frac{dy}{dt} + ay = b. \tag{19}$$

The derivation in Example 1 can now be repeated line for line, the only changes being that the coefficients 2 and 3 in Eq. (6) are replaced by a and b, respectively. The integrating factor is $\mu(t) = e^{at}$ and the solution is given by Eq. (5), which is the same as Eq. (18) with 2 replaced by a and 3 replaced by b.

The next stage is to replace the constant b by a given function $g(t)$, so that the differential equation becomes

$$\frac{dy}{dt} + ay = g(t). \tag{20}$$

The integrating factor depends only on the coefficient of y so for Eq. (20) the integrating factor is again $\mu(t) = e^{at}$. Multiplying Eq. (20) by $\mu(t)$, we obtain

$$e^{at}\frac{dy}{dt} + ae^{at}y = e^{at}g(t),$$

or

$$\frac{d}{dt}(e^{at}y) = e^{at}g(t). \tag{21}$$

By integrating both sides of Eq. (21) we find that

$$e^{at}y = \int e^{as}g(s)\,ds + c, \tag{22}$$

where c is an arbitrary constant. Note that we have used s to denote the integration variable to distinguish it from the independent variable t. By solving Eq. (22) for y we obtain the general solution

$$y = e^{-at}\int e^{as}g(s)\,ds + ce^{-at}. \tag{23}$$

For many simple functions $g(s)$ the integral in Eq. (23) can be evaluated and the solution y expressed in terms of elementary functions, as in the following examples. However, for more complicated functions $g(s)$, it may be necessary to leave the solution in the integral form given by Eq. (23).

EXAMPLE 2

Solve the differential equation

$$\frac{dy}{dt} + \tfrac{1}{2}y = 2 + t. \tag{24}$$

Sketch several solutions and find the particular solution whose graph contains the point (0, 2).

In this case $a = 1/2$, so the integrating factor is $\mu(t) = e^{t/2}$. Multiplying Eq. (24) by this factor leads to the equation

$$\frac{d}{dt}(e^{t/2}y) = 2e^{t/2} + te^{t/2}. \tag{25}$$

By integrating both sides of Eq. (25), using integration by parts on the second term on the right side, we obtain

$$e^{t/2}y = 4e^{t/2} + 2te^{t/2} - 4e^{t/2} + c,$$

where c is an arbitrary constant. Thus

$$y = 2t + ce^{-t/2}. \tag{26}$$

To find the solution that passes through the initial point (0, 2), we set $t = 0$ and $y = 2$ in Eq. (26), with the result that $2 = 0 + c$, so that $c = 2$. Hence the desired solution is

$$y = 2t + 2e^{-t/2}. \tag{27}$$

Graphs of the solution (26) for several values of c are shown in Figure 2.1.2. Observe that the solutions converge, not to a constant solution as in Example 1 and in the examples in Chapter 1, but to the solution $y = 2t$, which corresponds to $c = 0$.

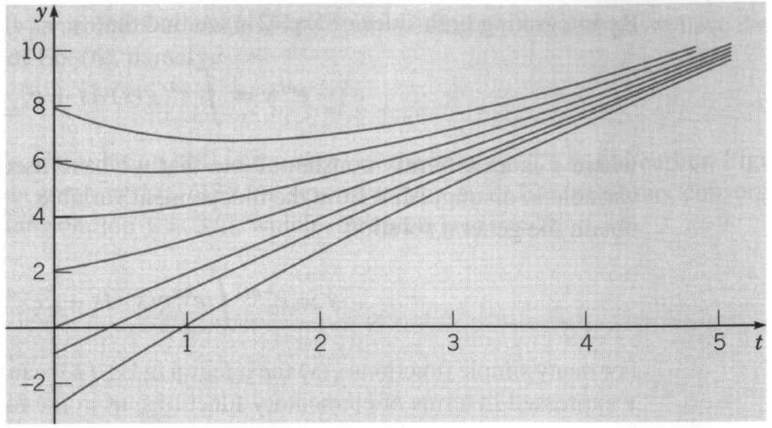

FIGURE 2.1.2 Integral curves of $y' + \frac{1}{2}y = 2 + t$.

EXAMPLE 3

Solve the differential equation

$$\frac{dy}{dt} - 2y = 4 - t \tag{28}$$

and sketch the graphs of several solutions. Find the initial point on the y-axis that separates solutions that grow large positively from those that grow large negatively as $t \to \infty$.

Since the coefficient of y is -2, the integrating factor for Eq. (28) is $\mu(t) = e^{-2t}$. Multiplying the differential equation by $\mu(t)$, we obtain

$$\frac{d}{dt}(e^{-2t}y) = 4e^{-2t} - te^{-2t}. \tag{29}$$

Then, by integrating both sides of this equation, we have

$$e^{-2t}y = -2e^{-2t} + \tfrac{1}{2}te^{-2t} + \tfrac{1}{4}e^{-2t} + c,$$

where we have used integration by parts on the last term in Eq. (29). Thus the general solution of Eq. (28) is

$$y = -\tfrac{7}{4} + \tfrac{1}{2}t + ce^{2t}. \tag{30}$$

Graphs of the solution (30) for several values of c are shown in Figure 2.1.3. The behavior of the solution for large values of t is determined by the term ce^{2t}. If $c \neq 0$, then the solution grows exponentially large in magnitude, with the same sign as c itself. Thus the solutions diverge as t becomes large. The boundary between solutions that ultimately grow positively from those that ultimately grow negatively occurs when $c = 0$. If we substitute $c = 0$ into Eq. (30) and then set $t = 0$, we find that $y = -7/4$. This is the separation point on the y-axis that was requested. Note that, for this initial value, the solution is $y = -\frac{7}{4} + \frac{1}{2}t$; it grows positively (but not exponentially).

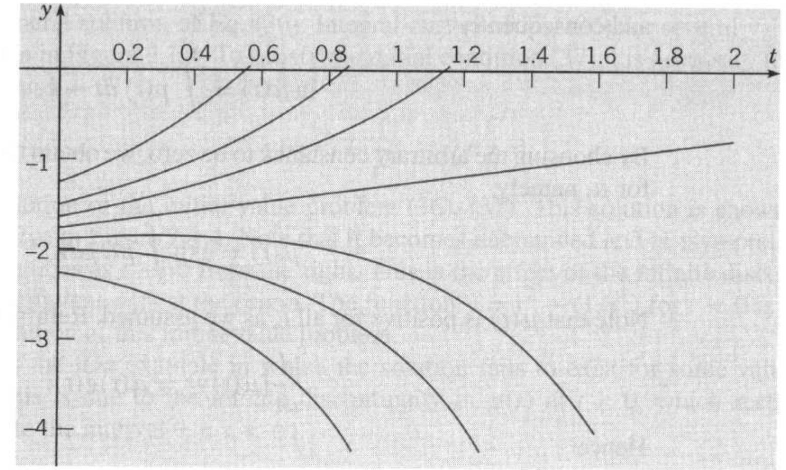

FIGURE 2.1.3 Integral curves of $y' - 2y = 4 - t$.

Examples 2 and 3 are special cases of Eq. (20),

$$\frac{dy}{dt} + ay = g(t),$$

whose solutions are given by Eq. (23),

$$y = e^{-at}\int e^{as}g(s)\,ds + ce^{-at}.$$

The solutions converge if $a > 0$, as in Example 2, and diverge if $a < 0$, as in Example 3. In contrast to the equations considered in Sections 1.1 and 1.2, however, Eq. (20) does not have an equilibrium solution.

The final stage in extending the method of integrating factors is to the general first order linear equation (3),

$$\frac{dy}{dt} + p(t)y = g(t),$$

where p and g are given functions. If we multiply Eq. (3) by an as yet undetermined function $\mu(t)$, we obtain

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t). \tag{31}$$

Following the same line of development as in Example 1, we see that the left side of Eq. (31) is the derivative of the product $\mu(t)y$, provided that $\mu(t)$ satisfies the equation

$$\frac{d\mu(t)}{dt} = p(t)\mu(t). \tag{32}$$

If we assume temporarily that $\mu(t)$ is positive, then we have

$$\frac{d\mu(t)/dt}{\mu(t)} = p(t),$$

and consequently

$$\ln \mu(t) = \int p(t)\,dt + k.$$

By choosing the arbitrary constant k to be zero, we obtain the simplest possible function for μ, namely,

$$\mu(t) = \exp \int p(t)\,dt. \tag{33}$$

Note that $\mu(t)$ is positive for all t, as we assumed. Returning to Eq. (31), we have

$$\frac{d}{dt}[\mu(t)y] = \mu(t)g(t). \tag{34}$$

Hence

$$\mu(t)y = \int \mu(s)g(s)\,ds + c,$$

so the general solution of Eq. (3) is

$$y = \frac{\int \mu(s)g(s)\,ds + c}{\mu(t)}. \tag{35}$$

Observe that, to find the solution given by Eq. (35), two integrations are required: one to obtain $\mu(t)$ from Eq. (33) and the other to obtain y from Eq. (35).

EXAMPLE 4

Solve the initial value problem

$$ty' + 2y = 4t^2, \tag{36}$$

$$y(1) = 2. \tag{37}$$

Rewriting Eq. (36) in the standard form (3), we have

$$y' + (2/t)y = 4t, \tag{38}$$

so $p(t) = 2/t$ and $g(t) = 4t$. To solve Eq. (38) we first compute the integrating factor $\mu(t)$:

$$\mu(t) = \exp \int \frac{2}{t}\,dt = e^{2\ln|t|} = t^2. \tag{39}$$

On multiplying Eq. (38) by $\mu(t) = t^2$, we obtain

$$t^2y' + 2ty = (t^2y)' = 4t^3,$$

and therefore

$$t^2y = t^4 + c,$$

where c is an arbitrary constant. It follows that

$$y = t^2 + \frac{c}{t^2} \tag{40}$$

is the general solution of Eq. (36). Integral curves of Eq. (36) for several values of c are shown in Figure 2.1.4. To satisfy the initial condition (37) it is necessary to choose $c = 1$; thus

$$y = t^2 + \frac{1}{t^2}, \qquad t > 0 \tag{41}$$

is the solution of the initial value problem (36), (37). This solution is shown by the heavy curve in Figure 2.1.4. Note that it becomes unbounded and is asymptotic to the positive y-axis as $t \to 0$ from the right. This is the effect of the infinite discontinuity in the coefficient $p(t)$ at the origin. The function $y = t^2 + (1/t^2)$ for $t < 0$ is not part of the solution of this initial value problem.

This is the first example in which the solution fails to exist for some values of t. Again, this is due to the infinite discontinuity in $p(t)$ at $t = 0$, which restricts the solution to the interval $0 < t < \infty$.

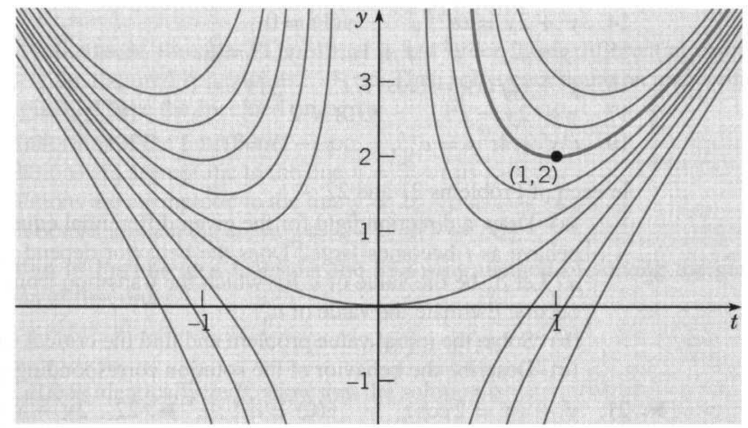

FIGURE 2.1.4 Integral curves of $ty' + 2y = 4t^2$.

Looking again at Figure 2.1.4, we see that some solutions (those for which $c > 0$) are asymptotic to the positive y-axis as $t \to 0$ from the right, while other solutions (for which $c < 0$) are asymptotic to the negative y-axis. The solution for which $c = 0$, namely, $y = t^2$, remains bounded and differentiable even at $t = 0$. If we generalize the initial condition (37) to

$$y(1) = y_0, \tag{42}$$

then $c = y_0 - 1$ and the solution (41) becomes

$$y = t^2 + \frac{y_0 - 1}{t^2}, \qquad t > 0. \tag{43}$$

As in Example 3, this is another instance where there is a critical initial value, namely, $y_0 = 1$, that separates solutions that behave in two quite different ways.

PROBLEMS

In each of Problems 1 through 12:

(a) Draw a direction field for the given differential equation.
(b) Based on an inspection of the direction field, describe how solutions behave for large t.
(c) Find the general solution of the given differential equation and use it to determine how solutions behave as $t \to \infty$.

▶ 1. $y' + 3y = t + e^{-2t}$
▶ 2. $y' - 2y = t^2 e^{2t}$
▶ 3. $y' + y = te^{-t} + 1$
▶ 4. $y' + (1/t)y = 3\cos 2t, \quad t > 0$
▶ 5. $y' - 2y = 3e^t$
▶ 6. $ty' + 2y = \sin t, \quad t > 0$
▶ 7. $y' + 2ty = 2te^{-t^2}$
▶ 8. $(1+t^2)y' + 4ty = (1+t^2)^{-2}$
▶ 9. $2y' + y = 3t$
▶ 10. $ty' - y = t^2 e^{-t}$
▶ 11. $y' + y = 5\sin 2t$
▶ 12. $2y' + y = 3t^2$

In each of Problems 13 through 20 find the solution of the given initial value problem.

13. $y' - y = 2te^{2t}, \quad y(0) = 1$
14. $y' + 2y = te^{-2t}, \quad y(1) = 0$
15. $ty' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}, \quad t > 0$
16. $y' + (2/t)y = (\cos t)/t^2 \quad y(\pi) = 0, \quad t > 0$
17. $y' - 2y = e^{2t}, \quad y(0) = 2$
18. $ty' + 2y = \sin t, \quad y(\pi/2) = 1$
19. $t^3 y' + 4t^2 y = e^{-t}, \quad y(-1) = 0$
20. $ty' + (t+1)y = t, \quad y(\ln 2) = 1$

In each of Problems 21 and 22:

(a) Draw a direction field for the given differential equation. How do solutions appear to behave as t becomes large? Does the behavior depend on the choice of the initial value a? Let a_0 be the value of a for which the transition from one type of behavior to another occurs. Estimate the value of a_0.
(b) Solve the initial value problem and find the critical value a_0 exactly.
(c) Describe the behavior of the solution corresponding to the initial value a_0.

▶ 21. $y' - \frac{1}{2}y = 2\cos t, \quad y(0) = a$
▶ 22. $2y' - y = e^{t/3}, \quad y(0) = a$

In each of Problems 23 and 24:

(a) Draw a direction field for the given differential equation. How do solutions appear to behave as $t \to 0$? Does the behavior depend on the choice of the initial value a? Let a_0 be the value of a for which the transition from one type of behavior to another occurs. Estimate the value of a_0.
(b) Solve the initial value problem and find the critical value a_0 exactly.
(c) Describe the behavior of the solution corresponding to the initial value a_0.

▶ 23. $ty' + (t+1)y = 2te^{-t}, \quad y(1) = a$
▶ 24. $ty' + 2y = (\sin t)/t, \quad y(-\pi/2) = a$

▶ 25. Consider the initial value problem

$$y' + \tfrac{1}{2}y = 2\cos t, \qquad y(0) = -1.$$

Find the coordinates of the first local maximum point of the solution for $t > 0$.

▶ 26. Consider the initial value problem

$$y' + \tfrac{2}{3}y = 1 - \tfrac{1}{2}t, \qquad y(0) = y_0.$$

Find the value of y_0 for which the solution touches, but does not cross, the t-axis.

▶ 27. Consider the initial value problem

$$y' + \tfrac{1}{4}y = 3 + 2\cos 2t, \qquad y(0) = 0.$$

(a) Find the solution of this initial value problem and describe its behavior for large t.
(b) Determine the value of t for which the solution first intersects the line $y = 12$.

28. Find the value of y_0 for which the solution of the initial value problem

$$y' - y = 1 + 3\sin t, \qquad y(0) = y_0$$

remains finite as $t \to \infty$.

29. Consider the initial value problem

$$y' - \tfrac{3}{2}y = 3t + 2e^t, \qquad y(0) = y_0.$$

Find the value of y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$?

30. Show that if a and λ are positive constants, and b is any real number, then every solution of the equation

$$y' + ay = be^{-\lambda t}$$

has the property that $y \to 0$ as $t \to \infty$.
Hint: Consider the cases $a = \lambda$ and $a \neq \lambda$ separately.

In each of Problems 31 through 34 construct a first order linear differential equation whose solutions have the required behavior as $t \to \infty$. Then solve your equation and confirm that the solutions do indeed have the specified property.

31. All solutions have the limit 3 as $t \to \infty$.
32. All solutions are asymptotic to the line $y = 3 - t$ as $t \to \infty$.
33. All solutions are asymptotic to the line $y = 2t - 5$ as $t \to \infty$.
34. All solutions approach the curve $y = 4 - t^2$ as $t \to \infty$.
35. **Variation of Parameters.** Consider the following method of solving the general linear equation of first order:

$$y' + p(t)y = g(t). \tag{i}$$

(a) If $g(t)$ is identically zero, show that the solution is

$$y = A\exp\left[-\int p(t)\,dt\right], \tag{ii}$$

where A is a constant.

(b) If $g(t)$ is not identically zero, assume that the solution is of the form

$$y = A(t)\exp\left[-\int p(t)\,dt\right], \tag{iii}$$

where A is now a function of t. By substituting for y in the given differential equation, show that $A(t)$ must satisfy the condition

$$A'(t) = g(t)\exp\left[\int p(t)\,dt\right]. \tag{iv}$$

(c) Find $A(t)$ from Eq. (iv). Then substitute for $A(t)$ in Eq. (iii) and determine y. Verify that the solution obtained in this manner agrees with that of Eq. (35) in the text. This technique is known as the method of **variation of parameters**; it is discussed in detail in Section 3.7 in connection with second order linear equations.

In each of Problems 36 and 37 use the method of Problem 35 to solve the given differential equation.

36. $y' - 2y = t^2 e^{2t}$
37. $y' + (1/t)y = 3\cos 2t, \quad t > 0$

2.2 Separable Equations

In Sections 1.2 and 2.1 we used a process of direct integration to solve first order linear equations of the form

$$\frac{dy}{dt} = ay + b, \tag{1}$$

where a and b are constants. We will now show that this process is actually applicable to a much larger class of equations.

We will use x to denote the independent variable in this section rather than t for two reasons. In the first place, different letters are frequently used for the variables in a differential equation, and you should not become too accustomed to using a single pair. In particular, x often occurs as the independent variable. Further, we want to reserve t for another purpose later in the section.

The general first order equation is

$$\frac{dy}{dx} = f(x, y). \tag{2}$$

Linear equations were considered in the preceding section, but if Eq. (2) is nonlinear, then there is no universally applicable method for solving the equation. Here, we consider a subclass of first order equations for which a direct integration process can be used.

To identify this class of equations we first rewrite Eq. (2) in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0. \tag{3}$$

It is always possible to do this by setting $M(x, y) = -f(x, y)$ and $N(x, y) = 1$, but there may be other ways as well. In the event that M is a function of x only and N is a function of y only, then Eq. (3) becomes

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{4}$$

Such an equation is said to be **separable**, because if it is written in the differential form

$$M(x)\,dx + N(y)\,dy = 0, \tag{5}$$

then, if you wish, terms involving each variable may be separated by the equals sign. The differential form (5) is also more symmetric and tends to diminish the distinction between independent and dependent variables.

EXAMPLE 1

Show that the equation

$$\frac{dy}{dx} = \frac{x^2}{1 - y^2} \tag{6}$$

is separable, and then find an equation for its integral curves.

If we write Eq. (6) as

$$-x^2 + (1 - y^2)\frac{dy}{dx} = 0, \tag{7}$$

then it has the form (4) and is therefore separable. Next, observe that the first term in Eq. (7) is the derivative of $-x^3/3$ and that the second term, by means of the chain rule, is the derivative with respect to x of $y - y^3/3$. Thus Eq. (7) can be written as

$$\frac{d}{dx}\left(-\frac{x^3}{3}\right) + \frac{d}{dx}\left(y - \frac{y^3}{3}\right) = 0,$$

or

$$\frac{d}{dx}\left(-\frac{x^3}{3} + y - \frac{y^3}{3}\right) = 0.$$

Therefore

$$-x^3 + 3y - y^3 = c, \tag{8}$$

where c is an arbitrary constant, is an equation for the integral curves of Eq. (6). A direction field and several integral curves are shown in Figure 2.2.1. An equation of the integral curve passing through a particular point (x_0, y_0) can be found by substituting x_0 and y_0 for x and y, respectively, in Eq. (8) and determining the corresponding value of c. Any differentiable function $y = \phi(x)$ that satisfies Eq. (8) is a solution of Eq. (6).

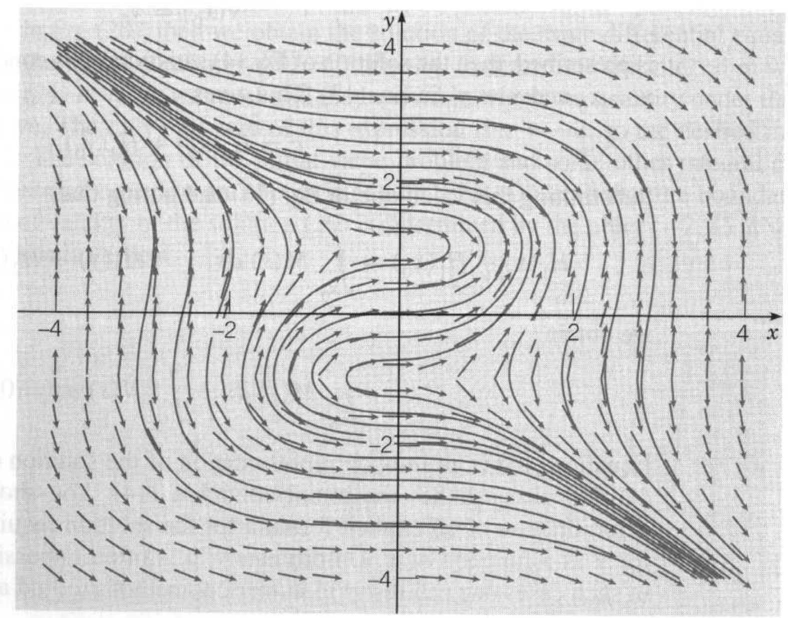

FIGURE 2.2.1 Direction field and integral curves of $y' = x^2/(1 - y^2)$.

Essentially the same procedure can be followed for any separable equation. Returning to Eq. (4), let H_1 and H_2 be any functions such that

$$H_1'(x) = M(x), \quad H_2'(y) = N(y); \tag{9}$$

then Eq. (4) becomes

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = 0. \tag{10}$$

According to the chain rule,

$$H_2'(y)\frac{dy}{dx} = \frac{d}{dx}H_2(y). \tag{11}$$

Consequently, Eq. (10) becomes

$$\frac{d}{dx}[H_1(x) + H_2(y)] = 0. \tag{12}$$

By integrating Eq. (12) we obtain

$$H_1(x) + H_2(y) = c, \tag{13}$$

where c is an arbitrary constant. Any differentiable function $y = \phi(x)$ that satisfies Eq. (13) is a solution of Eq. (4); in other words, Eq. (13) defines the solution implicitly rather than explicitly. The functions H_1 and H_2 are any antiderivatives of M and N, respectively. In practice, Eq. (13) is usually obtained from Eq. (5) by integrating the first term with respect to x and the second term with respect to y.

If, in addition to the differential equation, an initial condition

$$y(x_0) = y_0 \tag{14}$$

is prescribed, then the solution of Eq. (4) satisfying this condition is obtained by setting $x = x_0$ and $y = y_0$ in Eq. (13). This gives

$$c = H_1(x_0) + H_2(y_0). \tag{15}$$

Substituting this value of c in Eq. (13) and noting that

$$H_1(x) - H_1(x_0) = \int_{x_0}^{x} M(s)\,ds, \qquad H_2(y) - H_2(y_0) = \int_{y_0}^{y} N(s)\,ds,$$

we obtain

$$\int_{x_0}^{x} M(s)\,ds + \int_{y_0}^{y} N(s)\,ds = 0. \tag{16}$$

Equation (16) is an implicit representation of the solution of the differential equation (4) that also satisfies the initial condition (14). You should bear in mind that the determination of an explicit formula for the solution requires that Eq. (16) be solved for y as a function of x. Unfortunately, it is often impossible to do this analytically; in such cases one can resort to numerical methods to find approximate values of y for given values of x.

EXAMPLE 2

Solve the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \qquad y(0) = -1, \tag{17}$$

and determine the interval in which the solution exists.

The differential equation can be written as

$$2(y-1)\,dy = (3x^2 + 4x + 2)\,dx.$$

Integrating the left side with respect to y and the right side with respect to x gives

$$y^2 - 2y = x^3 + 2x^2 + 2x + c, \tag{18}$$

where c is an arbitrary constant. To determine the solution satisfying the prescribed initial condition, we substitute $x = 0$ and $y = -1$ in Eq. (18), obtaining $c = 3$. Hence the solution of the initial value problem is given implicitly by

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3. \tag{19}$$

To obtain the solution explicitly we must solve Eq. (19) for y in terms of x. This is a simple matter in this case, since Eq. (19) is quadratic in y, and we obtain

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}. \tag{20}$$

Equation (20) gives two solutions of the differential equation, only one of which, however, satisfies the given initial condition. This is the solution corresponding to the minus sign in Eq. (20), so that we finally obtain

$$y = \phi(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \tag{21}$$

as the solution of the initial value problem (17). Note that if the plus sign is chosen by mistake in Eq. (20), then we obtain the solution of the same differential equation that satisfies the initial condition $y(0) = 3$. Finally, to determine the interval in which the solution (21) is valid, we must find the interval in which the quantity under the radical is positive. The only real zero of this expression is $x = -2$, so the desired interval is $x > -2$. The solution of the initial value problem and some other integral curves of the differential equation are shown in Figure 2.2.2. Observe that the boundary of the interval of validity of the solution (20) is determined by the point $(-2, 1)$ at which the tangent line is vertical.

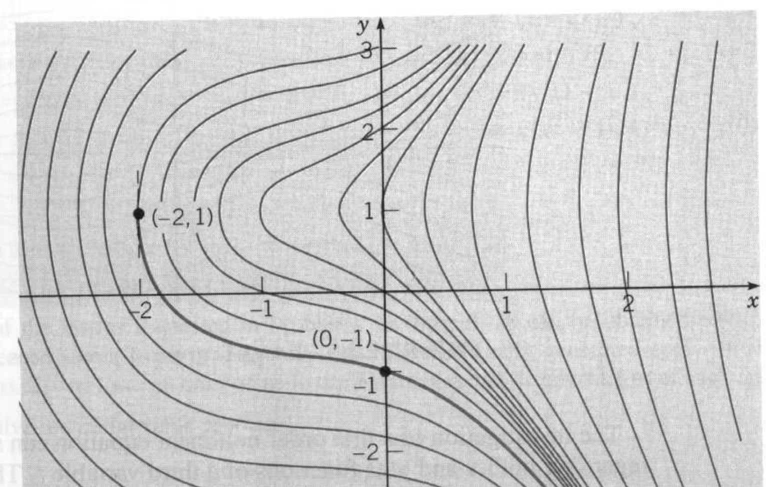

FIGURE 2.2.2 Integral curves of $y' = (3x^2 + 4x + 2)/2(y-1)$.

EXAMPLE 3

Find the solution of the initial value problem

$$\frac{dy}{dx} = \frac{y\cos x}{1+2y^2}, \qquad y(0) = 1. \tag{22}$$

Observe that $y = 0$ is a solution of the given differential equation. To find other solutions, assume that $y \neq 0$ and write the differential equation in the form

$$\frac{1+2y^2}{y}\,dy = \cos x\,dx. \tag{23}$$

Then, integrating the left side with respect to y and the right side with respect to x, we obtain

$$\ln|y| + y^2 = \sin x + c. \tag{24}$$

To satisfy the initial condition we substitute $x = 0$ and $y = 1$ in Eq. (24); this gives $c = 1$. Hence the solution of the initial value problem (22) is given implicitly by

$$\ln|y| + y^2 = \sin x + 1. \tag{25}$$

Since Eq. (25) is not readily solved for y as a function of x, further analysis of this problem becomes more delicate. One fairly evident fact is that no solution crosses the x-axis. To see this, observe that the left side of Eq. (25) becomes infinite if $y = 0$; however, the right side never becomes unbounded, so no point on the x-axis satisfies Eq. (25). Thus, for the solution of Eqs. (22) it follows that $y > 0$ always. Consequently, the absolute value bars in Eq. (25) can be dropped. It can also be shown that the interval of definition of the solution of the initial value problem (22) is the entire x-axis. Some integral curves of the given differential equation, including the solution of the initial value problem (22), are shown in Figure 2.2.3.

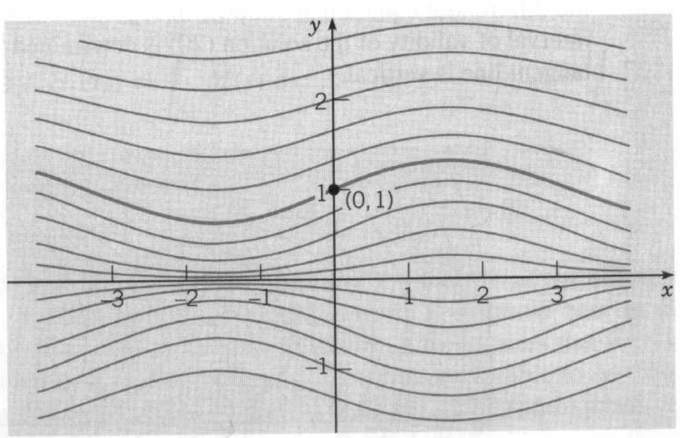

FIGURE 2.2.3 Integral curves of $y' = (y\cos x)/(1+2y^2)$.

The investigation of a first order nonlinear equation can sometimes be facilitated by regarding both x and y as functions of a third variable t. Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \tag{26}$$

If the differential equation is

$$\frac{dy}{dx} = \frac{F(x,y)}{G(x,y)}, \tag{27}$$

then, by comparing numerators and denominators in Eqs. (26) and (27), we obtain the system

$$dx/dt = G(x,y), \qquad dy/dt = F(x,y). \tag{28}$$

At first sight it may seem unlikely that a problem will be simplified by replacing a single equation by a pair of equations, but, in fact, the system (28) may well be more amenable to investigation than the single equation (27). Chapter 9 is devoted to nonlinear systems of the form (28).

Note: In Example 2 it was not difficult to solve explicitly for y as a function of x and to determine the exact interval in which the solution exists. However, this situation is exceptional, and often it will be better to leave the solution in implicit form, as in Examples 1 and 3. Thus, in the problems below and in other sections where nonlinear equations appear, the terminology "solve the following differential equation" means to find the solution explicitly if it is convenient to do so, but otherwise to find an implicit formula for the solution.

PROBLEMS

In each of Problems 1 through 8 solve the given differential equation.

1. $y' = x^2/y$
2. $y' = x^2/y(1+x^3)$
3. $y' + y^2\sin x = 0$
4. $y' = (3x^2-1)/(3+2y)$
5. $y' = (\cos^2 x)(\cos^2 2y)$
6. $xy' = (1-y^2)^{1/2}$
7. $\dfrac{dy}{dx} = \dfrac{x-e^{-x}}{y+e^y}$
8. $\dfrac{dy}{dx} = \dfrac{x^2}{1+y^2}$

In each of Problems 9 through 20:

(a) Find the solution of the given initial value problem in explicit form.
(b) Plot the graph of the solution.
(c) Determine (at least approximately) the interval in which the solution is defined.

▶ 9. $y' = (1-2x)y^2, \quad y(0) = -1/6$
▶ 10. $y' = (1-2x)/y, \quad y(1) = -2$
▶ 11. $x\,dx + ye^{-x}dy = 0, \quad y(0) = 1$
▶ 12. $dr/d\theta = r^2/\theta, \quad r(1) = 2$
▶ 13. $y' = 2x/(y+x^2y), \quad y(0) = -2$
▶ 14. $y' = xy^3(1+x^2)^{-1/2}, \quad y(0) = 1$
▶ 15. $y' = 2x/(1+2y), \quad y(2) = 0$
▶ 16. $y' = x(x^2+1)/4y^3, \quad y(0) = -1/\sqrt{2}$
▶ 17. $y' = (3x^2-e^x)/(2y-5), \quad y(0) = 1$
▶ 18. $y' = (e^{-x}-e^x)/(3+4y), \quad y(0) = 1$
▶ 19. $\sin 2x\,dx + \cos 3y\,dy = 0, \quad y(\pi/2) = \pi/3$
▶ 20. $y^2(1-x^2)^{1/2}dy = \arcsin x\,dx, \quad y(0) = 0$

Some of the results requested in Problems 21 through 28 can be obtained either by solving the given equations analytically, or by plotting numerically generated approximations to the solutions. Try to form an opinion as to the advantages and disadvantages of each approach.

▶ 21. Solve the initial value problem

$$y' = (1+3x^2)/(3y^2-6y), \qquad y(0) = 1$$

and determine the interval in which the solution is valid.
Hint: To find the interval of definition, look for points where the integral curve has a vertical

▶ 22. Solve the initial value problem

$$y' = 3x^2/(3y^2 - 4), \qquad y(1) = 0$$

and determine the interval in which the solution is valid.
Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

▶ 23. Solve the initial value problem

$$y' = 2y^2 + xy^2, \qquad y(0) = 1$$

and determine where the solution attains its minimum value.

▶ 24. Solve the initial value problem

$$y' = (2 - e^x)/(3 + 2y), \qquad y(0) = 0$$

and determine where the solution attains its maximum value.

▶ 25. Solve the initial value problem

$$y' = 2\cos 2x/(3 + 2y), \qquad y(0) = -1$$

and determine where the solution attains its maximum value.

▶ 26. Solve the initial value problem

$$y' = 2(1 + x)(1 + y^2), \qquad y(0) = 0$$

and determine where the solution attains its minimum value.

▶ 27. Consider the initial value problem

$$y' = ty(4 - y)/3, \qquad y(0) = y_0.$$

(a) Determine how the behavior of the solution as t increases depends on the initial value y_0.
(b) Suppose that $y_0 = 0.5$. Find the time T at which the solution first reaches the value 3.98.

▶ 28. Consider the initial value problem

$$y' = ty(4 - y)/(1 + t), \qquad y(0) = y_0 > 0.$$

(a) Determine how the solution behaves as $t \to \infty$.
(b) If $y_0 = 2$, find the time T at which the solution first reaches the value 3.99.
(c) Find the range of initial values for which the solution lies in the interval $3.99 < y < 4.01$ by the time $t = 2$.

29. Solve the equation

$$\frac{dy}{dx} = \frac{ay + b}{cy + d},$$

where a, b, c, and d are constants.

Homogeneous Equations. If the right side of the equation $dy/dx = f(x, y)$ can be expressed as a function of the ratio y/x only, then the equation is said to be homogeneous. Such equations can always be transformed into separable equations by a change of the dependent variable. Problem 30 illustrates how to solve first order homogeneous equations.

▶ 30. Consider the equation

$$\frac{dy}{dx} = \frac{y - 4x}{} \qquad \text{(i)}$$

(a) Show that Eq. (i) can be rewritten as

$$\frac{dy}{dx} = \frac{(y/x) - 4}{1 - (y/x)}; \qquad \text{(ii)}$$

thus Eq. (i) is homogeneous.
(b) Introduce a new dependent variable v so that $v = y/x$, or $y = xv(x)$. Express dy/dx in terms of x, v, and dv/dx.
(c) Replace y and dy/dx in Eq. (ii) by the expressions from part (b) that involve v and dv/dx. Show that the resulting differential equation is

$$v + x\frac{dv}{dx} = \frac{v - 4}{1 - v},$$

or

$$x\frac{dv}{dx} = \frac{v^2 - 4}{1 - v}. \qquad \text{(iii)}$$

Observe that Eq. (iii) is separable.
(d) Solve Eq. (iii) for v in terms of x.
(e) Find the solution of Eq. (i) by replacing v by y/x in the solution in part (d).
(f) Draw a direction field and some integral curves for Eq. (i). Recall that the right side of Eq. (1) actually depends only on the ratio y/x. This means that integral curves have the same slope at all points on any given straight line through the origin, although the slope changes from one line to another. Therefore the direction field and the integral curves are symmetric with respect to the origin. Is this symmetry property evident from your plot?

The method outlined in Problem 30 can be used for any homogeneous equation. That is, the substitution $y = xv(x)$ transforms a homogeneous equation into a separable equation. The latter equation can be solved by direct integration, and then replacing v by y/x gives the solution to the original equation. In each of Problems 31 through 38:

(a) Show that the given equation is homogeneous.
(b) Solve the differential equation.
(c) Draw a direction field and some integral curves. Are they symmetric with respect to the origin?

▶ 31. $\dfrac{dy}{dx} = \dfrac{x^2 + xy + y^2}{x^2}$

▶ 32. $\dfrac{dy}{dx} = \dfrac{x^2 + 3y^2}{2xy}$

▶ 33. $\dfrac{dy}{dx} = \dfrac{4y - 3x}{2x - y}$

▶ 34. $\dfrac{dy}{dx} = -\dfrac{4x + 3y}{2x + y}$

▶ 35. $\dfrac{dy}{dx} = \dfrac{x + 3y}{x - y}$

▶ 36. $(x^2 + 3xy + y^2)\,dx - x^2 dy = 0$

▶ 37. $\dfrac{dy}{dx} = \dfrac{x^2 - 3y^2}{2xy}$

▶ 38. $\dfrac{dy}{dx} = \dfrac{3y^2 - x^2}{2xy}$

2.3 Modeling with First Order Equations

Differential equations are of interest to nonmathematicians primarily because of the possibility of using them to investigate a wide variety of problems in the physical, biological, and social sciences. One reason for this is that mathematical models and