the proportion of susceptibles and carriers in the population. Suppose that carriers are identified and removed from the population at a rate β, so

$$dy/dt = -\beta y. \qquad \text{(i)}$$

Suppose also that the disease spreads at a rate proportional to the product of x and y; thus

$$dx/dt = \alpha xy. \qquad \text{(ii)}$$

(a) Determine y at any time t by solving Eq. (i) subject to the initial condition $y(0) = y_0$.
(b) Use the result of part (a) to find x at any time t by solving Eq. (ii) subject to the initial condition $x(0) = x_0$.
(c) Find the proportion of the population that escapes the epidemic by finding the limiting value of x as $t \to \infty$.

24. Daniel Bernoulli's work in 1760 had the goal of appraising the effectiveness of a controversial inoculation program against smallpox, which at that time was a major threat to public health. His model applies equally well to any other disease that, once contracted and survived, confers a lifetime immunity.

Consider the cohort of individuals born in a given year ($t = 0$), and let $n(t)$ be the number of these individuals surviving t years later. Let $x(t)$ be the number of members of this cohort who have not had smallpox by year t, and who are therefore still susceptible. Let β be the rate at which susceptibles contract smallpox, and let ν be the rate at which people who contract smallpox die from the disease. Finally, let $\mu(t)$ be the death rate from all causes other than smallpox. Then dx/dt, the rate at which the number of susceptibles declines, is given by

$$dx/dt = -[\beta + \mu(t)]x; \qquad \text{(i)}$$

the first term on the right side of Eq. (i) is the rate at which susceptibles contract smallpox, while the second term is the rate at which they die from all other causes. Also

$$dn/dt = -\nu\beta x - \mu(t)n, \qquad \text{(ii)}$$

where dn/dt is the death rate of the entire cohort, and the two terms on the right side are the death rates due to smallpox and to all other causes, respectively.
(a) Let $z = x/n$ and show that z satisfies the initial value problem

$$dz/dt = -\beta z(1 - \nu z), \qquad z(0) = 1. \qquad \text{(iii)}$$

Observe that the initial value problem (iii) does not depend on $\mu(t)$.
(b) Find $z(t)$ by solving Eq. (iii).
(c) Bernoulli estimated that $\nu = \beta = \frac{1}{8}$. Using these values, determine the proportion of 20-year-olds who have not had smallpox.

Note: Based on the model just described and using the best mortality data available at the time, Bernoulli calculated that if deaths due to smallpox could be eliminated ($\nu = 0$), then approximately 3 years could be added to the average life expectancy (in 1760) of 26 years 7 months. He therefore supported the inoculation program.

25. **Bifurcation Points.** In many physical problems some observable quantity, such as a velocity, waveform, or chemical reaction, depends on a parameter describing the physical state. As this parameter is increased, a critical value is reached at which the velocity, or waveform, or reaction suddenly changes its character. For example, as the amount of one of the chemicals in a certain mixture is increased, spiral wave patterns of varying color suddenly emerge in an originally quiescent fluid. In many such cases the mathematical analysis ultimately leads to an equation[10] of the form

$$dx/dt = (R - R_c)x - ax^3. \qquad \text{(i)}$$

Here a and R_c are positive constants, and R is a parameter that may take on various values. For example, R may measure the amount of a certain chemical and x may measure a chemical reaction.
(a) If $R < R_c$, show that there is only one equilibrium solution $x = 0$ and that it is asymptotically stable.
(b) If $R > R_c$, show that there are three equilibrium solutions, $x = 0$ and $x = \pm\sqrt{(R - R_c)/a}$, and that the first solution is unstable while the other two are asymptotically stable.
(c) Draw a graph in the Rx-plane showing all equilibrium solutions and label each one as asymptotically stable or unstable.

The point $R = R_c$ is called a **bifurcation point**. For $R < R_c$ one observes the asymptotically stable equilibrium solution $x = 0$. However, this solution loses its stability as R passes through the value R_c, and for $R > R_c$ the asymptotically stable (and hence the observable) solutions are $x = \sqrt{(R - R_c)/a}$ and $x = -\sqrt{(R - R_c)/a}$. Because of the way in which the solutions branch at R_c, this type of bifurcation is called a pitchfork bifurcation; your sketch should suggest that this name is appropriate.

26. **Chemical Reactions.** A second order chemical reaction involves the interaction (collision) of one molecule of a substance P with one molecule of a substance Q to produce one molecule of a new substance X; this is denoted by $P + Q \to X$. Suppose that p and q, where $p \neq q$, are the initial concentrations of P and Q, respectively, and let $x(t)$ be the concentration of X at time t. Then $p - x(t)$ and $q - x(t)$ are the concentrations of P and Q at time t, and the rate at which the reaction occurs is given by the equation

$$dx/dt = \alpha(p - x)(q - x), \qquad \text{(i)}$$

where α is a positive constant.
(a) If $x(0) = 0$, determine the limiting value of $x(t)$ as $t \to \infty$ without solving the differential equation. Then solve the initial value problem and find $x(t)$ for any t.
(b) If the substances P and Q are the same, then $p = q$ and Eq. (i) is replaced by

$$dx/dt = \alpha(p - x)^2. \qquad \text{(ii)}$$

If $x(0) = 0$, determine the limiting value of $x(t)$ as $t \to \infty$ without solving the differential equation. Then solve the initial value problem and determine $x(t)$ for any t.

2.6 Exact Equations and Integrating Factors

For first order equations there are a number of integration methods that are applicable to various classes of problems. The most important of these are linear equations and separable equations, which we have discussed previously. Here, we consider a class of

[10] In fluid mechanics Eq. (i) arises in the study of the transition from laminar to turbulent flow; there it is often called the Landau equation. L. D. Landau (1908–1968) was a Russian physicist who received the Nobel prize in 1962 for his contributions to the understanding of condensed states, particularly liquid helium. He was also the coauthor, with E. M. Lifschitz, of a well-known series of physics textbooks.

equations known as exact equations for which there is also a well-defined method of solution. Keep in mind, however, that those first order equations that can be solved by elementary integration methods are rather special; most first order equations cannot be solved in this way.

EXAMPLE 1

Solve the differential equation

$$2x + y^2 + 2xyy' = 0. \tag{1}$$

The equation is neither linear nor separable, so the methods suitable for those types of equations are not applicable here. However, observe that the function $\psi(x, y) = x^2 + xy^2$ has the property that

$$2x + y^2 = \frac{\partial \psi}{\partial x}, \qquad 2xy = \frac{\partial \psi}{\partial y}. \tag{2}$$

Therefore the differential equation can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

Assuming that y is a function of x and calling upon the chain rule, we can write Eq. (3) in the equivalent form

$$\frac{d\psi}{dx} = \frac{d}{dx}(x^2 + xy^2) = 0. \tag{4}$$

Therefore

$$\psi(x, y) = x^2 + xy^2 = c, \tag{5}$$

where c is an arbitrary constant, is an equation that defines solutions of Eq. (1) implicitly.

In solving Eq. (1) the key step was the recognition that there is a function ψ that satisfies Eq. (2). More generally, let the differential equation

$$M(x, y) + N(x, y)y' = 0 \tag{6}$$

be given. Suppose that we can identify a function ψ such that

$$\frac{\partial \psi}{\partial x}(x, y) = M(x, y), \qquad \frac{\partial \psi}{\partial y}(x, y) = N(x, y), \tag{7}$$

and such that $\psi(x, y) = c$ defines $y = \phi(x)$ implicitly as a differentiable function of x. Then

$$M(x, y) + N(x, y)y' = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\psi[x, \phi(x)]$$

and the differential equation (6) becomes

$$\frac{d}{dx}\psi[x, \phi(x)] = 0. \tag{8}$$

In this case Eq. (6) is said to be an **exact** differential equation. Solutions of Eq. (6), or the equivalent Eq. (8), are given implicitly by

$$\psi(x, y) = c, \tag{9}$$

where c is an arbitrary constant.

In Example 1 it was relatively easy to see that the differential equation was exact and, in fact, easy to find its solution, by recognizing the required function ψ. For more complicated equations it may not be possible to do this so easily. A systematic way of determining whether a given differential equation is exact is provided by the following theorem.

Theorem 2.6.1 Let the functions M, N, M_y, and N_x, where subscripts denote partial derivatives, be continuous in the rectangular[11] region $R: \alpha < x < \beta, \gamma < y < \delta$. Then Eq. (6),

$$M(x, y) + N(x, y)y' = 0,$$

is an exact differential equation in R if and only if

$$M_y(x, y) = N_x(x, y) \tag{10}$$

at each point of R. That is, there exists a function ψ satisfying Eqs. (7),

$$\psi_x(x, y) = M(x, y), \qquad \psi_y(x, y) = N(x, y),$$

if and only if M and N satisfy Eq. (10).

The proof of this theorem has two parts. First, we show that if there is a function ψ such that Eqs. (7) are true, then it follows that Eq. (10) is satisfied. Computing M_y and N_x from Eqs. (7), we obtain

$$M_y(x, y) = \psi_{xy}(x, y), \qquad N_x(x, y) = \psi_{yx}(x, y). \tag{11}$$

Since M_y and N_x are continuous, it follows that ψ_{xy} and ψ_{yx} are also continuous. This guarantees their equality, and Eq. (10) follows.

We now show that if M and N satisfy Eq. (10), then Eq. (6) is exact. The proof involves the construction of a function ψ satisfying Eqs. (7),

$$\psi_x(x, y) = M(x, y), \qquad \psi_y(x, y) = N(x, y).$$

Integrating the first of Eqs. (7) with respect to x, holding y constant, we find that

$$\psi(x, y) = \int M(x, y)\,dx + h(y). \tag{12}$$

The function h is an arbitrary function of y, playing the role of the arbitrary constant. Now we must show that it is always possible to choose $h(y)$ so that $\psi_y = N$. From Eq. (12)

$$\begin{aligned}\psi_y(x, y) &= \frac{\partial}{\partial y}\int M(x, y)\,dx + h'(y)\\ &= \int M_y(x, y)\,dx + h'(y).\end{aligned}$$

[11]It is not essential that the region be rectangular, only that it be simply connected. In two dimensions this means that the region has no holes in its interior. Thus, for example, rectangular or circular regions are simply connected, but an annular region is not. More details can be found in most books on advanced calculus.

Setting $\psi_y = N$ and solving for $h'(y)$, we obtain

$$h'(y) = N(x, y) - \int M_y(x, y)\,dx. \tag{13}$$

To determine $h(y)$ from Eq. (13), it is essential that, despite its appearance, the right side of Eq. (13) be a function of y only. To establish this fact, we can differentiate the quantity in question with respect to x, obtaining

$$N_x(x, y) - M_y(x, y),$$

which is zero on account of Eq. (10). Hence, despite its apparent form, the right side of Eq. (13) does not, in fact, depend on x, and a single integration then gives $h(y)$. Substituting for $h(y)$ in Eq. (12), we obtain as the solution of Eqs. (7)

$$\psi(x, y) = \int M(x, y)\,dx + \int \left[N(x, y) - \int M_y(x, y)\,dx\right] dy. \tag{14}$$

It should be noted that this proof contains a method for the computation of $\psi(x, y)$ and thus for solving the original differential equation (6). It is usually better to go through this process each time it is needed rather than to try to remember the result given in Eq. (14). Note also that the solution is obtained in implicit form; it may or may not be feasible to find the solution explicitly.

EXAMPLE 2

Solve the differential equation

$$(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0. \tag{15}$$

It is easy to see that

$$M_y(x, y) = \cos x + 2xe^y = N_x(x, y),$$

so the given equation is exact. Thus there is a $\psi(x, y)$ such that

$$\psi_x(x, y) = y\cos x + 2xe^y,$$
$$\psi_y(x, y) = \sin x + x^2e^y - 1.$$

Integrating the first of these equations, we obtain

$$\psi(x, y) = y\sin x + x^2e^y + h(y). \tag{16}$$

Setting $\psi_y = N$ gives

$$\psi_y(x, y) = \sin x + x^2e^y + h'(y) = \sin x + x^2e^y - 1.$$

Thus $h'(y) = -1$ and $h(y) = -y$. The constant of integration can be omitted since any solution of the preceding differential equation is satisfactory; we do not require the most general one. Substituting for $h(y)$ in Eq. (16) gives

$$\psi(x, y) = y\sin x + x^2e^y - y.$$

Hence solutions of Eq. (15) are given implicitly by

$$y\sin x + x^2e^y - y = c. \tag{17}$$

EXAMPLE 3

Solve the differential equation

$$(3xy + y^2) + (x^2 + xy)y' = 0. \tag{18}$$

Here,

$$M_y(x, y) = 3x + 2y, \qquad N_x(x, y) = 2x + y;$$

since $M_y \neq N_x$, the given equation is not exact. To see that it cannot be solved by the procedure described previously, let us seek a function ψ such that

$$\psi_x(x, y) = 3xy + y^2, \qquad \psi_y(x, y) = x^2 + xy. \tag{19}$$

Integrating the first of Eqs. (19) gives

$$\psi(x,y) = \tfrac{3}{2}x^2y + xy^2 + h(y), \tag{20}$$

where h is an arbitrary function of y only. To try to satisfy the second of Eqs. (19) we compute ψ_y from Eq. (20) and set it equal to N, obtaining

$$\tfrac{3}{2}x^2 + 2xy + h'(y) = x^2 + xy$$

or

$$h'(y) = -\tfrac{1}{2}x^2 - xy. \tag{21}$$

Since the right side of Eq. (21) depends on x as well as y, it is impossible to solve Eq. (21) for $h(y)$. Thus there is no $\psi(x, y)$ satisfying both of Eqs. (19).

Integrating Factors. It is sometimes possible to convert a differential equation that is not exact into an exact equation by multiplying the equation by a suitable integrating factor. Recall that this is the procedure that we used in solving linear equations in Section 2.1. To investigate the possibility of implementing this idea more generally, let us multiply the equation

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{22}$$

by a function μ and then try to choose μ so that the resulting equation

$$\mu(x, y)M(x, y)\,dx + \mu(x, y)N(x, y)\,dy = 0 \tag{23}$$

is exact. By Theorem 2.6.1 Eq. (23) is exact if and only if

$$(\mu M)_y = (\mu N)_x. \tag{24}$$

Since M and N are given functions, Eq. (24) states that the integrating factor μ must satisfy the first order partial differential equation

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0. \tag{25}$$

If a function μ satisfying Eq. (25) can be found, then Eq. (23) will be exact. The solution of Eq. (23) can then be obtained by the method described in the first part of this section. The solution found in this way also satisfies Eq. (22), since the integrating factor μ can be canceled out of Eq. (23).

A partial differential equation of the form (25) may have more than one solution; if this is the case, any such solution may be used as an integrating factor of Eq. (22). This possible nonuniqueness of the integrating factor is illustrated in Example 4.

Unfortunately, Eq. (25), which determines the integrating factor μ, is ordinarily at least as difficult to solve as the original equation (22). Therefore, while in principle integrating factors are powerful tools for solving differential equations, in practice they can be found only in special cases. The most important situations in which simple integrating factors can be found occur when μ is a function of only one of the variables x or y, instead of both. Let us determine necessary conditions on M and N so that Eq. (22) has an integrating factor μ that depends on x only. Assuming that μ is a function of x only, we have

$$(\mu M)_y = \mu M_y, \qquad (\mu N)_x = \mu N_x + N\frac{d\mu}{dx}.$$

Thus, if $(\mu M)_y$ is to equal $(\mu N)_x$, it is necessary that

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu. \tag{26}$$

If $(M_y - N_x)/N$ is a function of x only, then there is an integrating factor μ that also depends only on x; further, $\mu(x)$ can be found by solving Eq. (26), which is both linear and separable.

A similar procedure can be used to determine a condition under which Eq. (22) has an integrating factor depending only on y; see Problem 23.

EXAMPLE 4

Find an integrating factor for the equation

$$(3xy + y^2) + (x^2 + xy)y' = 0 \tag{18}$$

and then solve the equation.

In Example 3 we showed that this equation is not exact. Let us determine whether it has an integrating factor that depends on x only. On computing the quantity $(M_y - N_x)/N$, we find that

$$\frac{M_y(x,y) - N_x(x,y)}{N(x,y)} = \frac{3x + 2y - (2x + y)}{x^2 + xy} = \frac{1}{x}. \tag{27}$$

Thus there is an integrating factor μ that is a function of x only, and it satisfies the differential equation

$$\frac{d\mu}{dx} = \frac{\mu}{x}. \tag{28}$$

Hence

$$\mu(x) = x. \tag{29}$$

Multiplying Eq. (18) by this integrating factor, we obtain

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0. \tag{30}$$

The latter equation is exact and it is easy to show that its solutions are given implicitly by

$$x^3y + \tfrac{1}{2}x^2y^2 = c. \tag{31}$$

Solutions may also be readily found in explicit form since Eq. (31) is quadratic in y.

You may also verify that a second integrating factor of Eq. (18) is

$$\mu(x,y) = \frac{1}{xy(2x+y)},$$

and that the same solution is obtained, though with much greater difficulty, if this integrating factor is used (see Problem 32).

PROBLEMS

Determine whether or not each of the equations in Problems 1 through 12 is exact. If it is exact, find the solution.

1. $(2x + 3) + (2y - 2)y' = 0$
2. $(2x + 4y) + (2x - 2y)y' = 0$
3. $(3x^2 - 2xy + 2)\,dx + (6y^2 - x^2 + 3)\,dy = 0$
4. $(2xy^2 + 2y) + (2x^2y + 2x)y' = 0$
5. $\dfrac{dy}{dx} = -\dfrac{ax + by}{bx + cy}$
6. $\dfrac{dy}{dx} = -\dfrac{ax - by}{bx - cy}$
7. $(e^x \sin y - 2y \sin x)\,dx + (e^x \cos y + 2\cos x)\,dy = 0$
8. $(e^x \sin y + 3y)\,dx - (3x - e^x \sin y)\,dy = 0$
9. $(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x)\,dx + (xe^{xy}\cos 2x - 3)\,dy = 0$
10. $(y/x + 6x)\,dx + (\ln x - 2)\,dy = 0, \quad x > 0$
11. $(x \ln y + xy)\,dx + (y \ln x + xy)\,dy = 0; \quad x > 0, \quad y > 0$
12. $\dfrac{x\,dx}{(x^2 + y^2)^{3/2}} + \dfrac{y\,dy}{(x^2 + y^2)^{3/2}} = 0$

In each of Problems 13 and 14 solve the given initial value problem and determine at least approximately where the solution is valid.

13. $(2x - y)\,dx + (2y - x)\,dy = 0, \quad y(1) = 3$
14. $(9x^2 + y - 1)\,dx - (4y - x)\,dy = 0, \quad y(1) = 0$

In each of Problems 15 and 16 find the value of b for which the given equation is exact and then solve it using that value of b.

15. $(xy^2 + bx^2y)\,dx + (x + y)x^2\,dy = 0$
16. $(ye^{2xy} + x)\,dx + bxe^{2xy}\,dy = 0$
17. Consider the exact differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

Find an implicit formula $\psi(x,y) = c$ for the solution analogous to Eq. (14) by first integrating the equation $\psi_y = N$, rather than $\psi_x = M$, as in the text.

18. Show that any separable equation,

$$M(x) + N(y)y' = 0,$$

is also exact.

Show that the equations in Problems 19 through 22 are not exact, but become exact when multiplied by the given integrating factor. Then solve the equations.

19. $x^2y^3 + x(1 + y^2)y' = 0, \quad \mu(x,y) = 1/xy^3$
20. $\left(\dfrac{\sin y}{y} - 2e^{-x}\sin x\right)dx + \left(\dfrac{\cos y + 2e^{-x}\cos x}{y}\right)dy = 0, \quad \mu(x,y) = ye^x$
21. $y\,dx + (2x - ye^y)\,dy = 0, \quad \mu(x,y) = y$
22. $(x + 2)\sin y\,dx + x\cos y\,dy = 0, \quad \mu(x,y) = xe^x$

23. Show that if $(N_x - M_y)/M = Q$, where Q is a function of y only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form

$$\mu(y) = \exp \int Q(y)\, dy.$$

24. Show that if $(N_x - M_y)/(xM - yN) = R$, where R depends on the quantity xy only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form $\mu(xy)$. Find a general formula for this integrating factor.

In each of Problems 25 through 31 find an integrating factor and solve the given equation.

25. $(3x^2y + 2xy + y^3)\, dx + (x^2 + y^2)\, dy = 0$
26. $y' = e^{2x} + y - 1$
27. $dx + (x/y - \sin y)\, dy = 0$
28. $y\, dx + (2xy - e^{-2y})\, dy = 0$
29. $e^x\, dx + (e^x \cot y + 2y \csc y)\, dy = 0$
30. $[4(x^3/y^2) + (3/y)]\, dx + [3(x/y^2) + 4y]\, dy = 0$
31. $\left(3x + \dfrac{6}{y}\right) + \left(\dfrac{x^2}{y} + 3\dfrac{y}{x}\right)\dfrac{dy}{dx} = 0$

    Hint: See Problem 24.

32. Solve the differential equation

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

using the integrating factor $\mu(x, y) = [xy(2x + y)]^{-1}$. Verify that the solution is the same as that obtained in Example 4 with a different integrating factor.

2.7 Numerical Approximations: Euler's Method

Recall two important facts about the first order initial value problem

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0. \tag{1}$$

First, if f and $\partial f/\partial y$ are continuous, then the initial value problem (1) has a unique solution $y = \phi(t)$ in some interval surrounding the initial point $t = t_0$. Second, it is usually not possible to find the solution ϕ by symbolic manipulations of the differential equation. Up to now we have considered the main exceptions to this statement, namely, differential equations that are linear, separable, or exact, or that can be transformed into one of these types. Nevertheless, it remains true that solutions of the vast majority of first order initial value problems cannot be found by analytical means such as those considered in the first part of this chapter.

Therefore it is important to be able to approach the problem in other ways. As we have already seen, one of these ways is to draw a direction field for the differential equation (which does not involve solving the equation) and then to visualize the behavior of solutions from the direction field. This has the advantage of being a relatively simple process, even for complicated differential equations. However, it does not lend itself to quantitative computations or comparisons, and this is often a critical shortcoming.

Another alternative is to compute approximate values of the solution $y = \phi(t)$ of the initial value problem (1) at a selected set of t-values. Ideally, the approximate solution values will be accompanied by error bounds that assure the level of accuracy of the approximations. Today there are numerous methods that produce numerical approximations to solutions of differential equations, and Chapter 8 is devoted to a fuller discussion of some of them. Here, we introduce the oldest and simplest such method, originated by Euler about 1768. It is called the **tangent line method** or the **Euler method**.

Let us consider how we might approximate the solution $y = \phi(t)$ of Eqs. (1) near $t = t_0$. We know that the solution passes through the initial point (t_0, y_0) and, from the differential equation, we also know that its slope at this point is $f(t_0, y_0)$. Thus we can write down an equation for the line tangent to the solution curve at (t_0, y_0), namely,

$$y = y_0 + f(t_0, y_0)(t - t_0). \tag{2}$$

The tangent line is a good approximation to the actual solution curve on an interval short enough so that the slope of the solution does not change appreciably from its value at the initial point; see Figure 2.7.1. Thus, if t_1 is close enough to t_0, we can approximate $\phi(t_1)$ by the value y_1 determined by substituting $t = t_1$ into the tangent line approximation at $t = t_0$; thus

$$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0). \tag{3}$$

To proceed further, we can try to repeat the process. Unfortunately, we do not know the value $\phi(t_1)$ of the solution at t_1. The best we can do is to use the approximate value y_1 instead. Thus we construct the line through (t_1, y_1) with the slope $f(t_1, y_1)$,

$$y = y_1 + f(t_1, y_1)(t - t_1). \tag{4}$$

To approximate the value of $\phi(t)$ at a nearby point t_2, we use Eq. (4) instead, obtaining

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1). \tag{5}$$

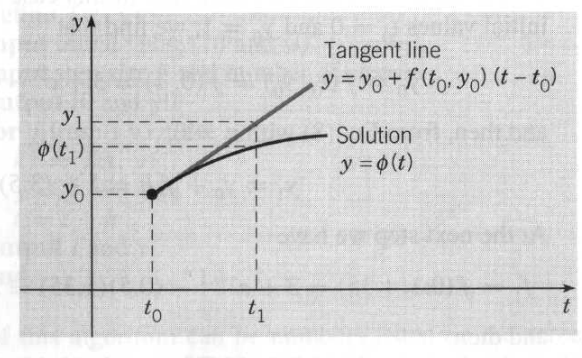

FIGURE 2.7.1 A tangent line approximation.

CHAPTER 3

Second Order Linear Equations

Linear equations of second order are of crucial importance in the study of differential equations for two main reasons. The first is that linear equations have a rich theoretical structure that underlies a number of systematic methods of solution. Further, a substantial portion of this structure and these methods are understandable at a fairly elementary mathematical level. In order to present the key ideas in the simplest possible context, we describe them in this chapter for second order equations. Another reason to study second order linear equations is that they are vital to any serious investigation of the classical areas of mathematical physics. One cannot go very far in the development of fluid mechanics, heat conduction, wave motion, or electromagnetic phenomena without finding it necessary to solve second order linear differential equations. As an example, we discuss the oscillations of some basic mechanical and electrical systems at the end of the chapter.

3.1 Homogeneous Equations with Constant Coefficients

A second order ordinary differential equation has the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right), \tag{1}$$

where f is some given function. Usually, we will denote the independent variable by t since time is often the independent variable in physical problems, but sometimes we will use x instead. We will use y, or occasionally some other letter, to designate the dependent variable. Equation (1) is said to be **linear** if the function f has the form

$$f\left(t, y, \frac{dy}{dt}\right) = g(t) - p(t)\frac{dy}{dt} - q(t)y, \tag{2}$$

that is, if f is linear in y and y'. In Eq. (2) g, p, and q are specified functions of the independent variable t but do not depend on y. In this case we usually rewrite Eq. (1) as

$$y'' + p(t)y' + q(t)y = g(t), \tag{3}$$

where the primes denote differentiation with respect to t. Instead of Eq. (3), we often see the equation

$$P(t)y'' + Q(t)y' + R(t)y = G(t). \tag{4}$$

Of course, if $P(t) \neq 0$, we can divide Eq. (4) by $P(t)$ and thereby obtain Eq. (3) with

$$p(t) = \frac{Q(t)}{P(t)}, \qquad q(t) = \frac{R(t)}{P(t)}, \qquad g(t) = \frac{G(t)}{P(t)}. \tag{5}$$

In discussing Eq. (3) and in trying to solve it, we will restrict ourselves to intervals in which p, q, and g are continuous functions.[1]

If Eq. (1) is not of the form (3) or (4), then it is called **nonlinear**. Analytical investigations of nonlinear equations are relatively difficult, so we will have little to say about them in this book. Numerical or geometical approaches are often more appropriate, and these are discussed in Chapters 8 and 9. In addition, there are two special types of second order nonlinear equations that can be solved by a change of variables that reduces them to first order equations. This procedure is outlined in Problems 28 through 43.

An initial value problem consists of a differential equation such as Eq. (1), (3), or (4) together with a pair of initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y_0', \tag{6}$$

where y_0 and y_0' are given numbers. Observe that the initial conditions for a second order equation prescribe not only a particular point (t_0, y_0) through which the graph of the solution must pass, but also the slope y_0' of the graph at that point. It is reasonable to expect that two initial conditions are needed for a second order equation because, roughly speaking, two integrations are required to find a solution and each integration introduces an arbitrary constant. Presumably, two initial conditions will suffice to determine values for these two constants.

A second order linear equation is said to be **homogeneous** if the term $g(t)$ in Eq. (3), or the term $G(t)$ in Eq. (4), is zero for all t. Otherwise, the equation is called **nonhomogeneous**. As a result, the term $g(t)$, or $G(t)$, is sometimes called the nonhomogeneous term. We begin our discussion with homogeneous equations, which we will write in the form

$$P(t)y'' + Q(t)y' + R(t)y = 0. \tag{7}$$

[1] There is a corresponding treatment of higher order linear equations in Chapter 4. If you wish, you may read the appropriate parts of Chapter 4 in parallel with Chapter 3.

Later, in Sections 3.6 and 3.7, we will show that once the homogeneous equation has been solved, it is always possible to solve the corresponding nonhomogeneous equation (4), or at least to express the solution in terms of an integral. Thus the problem of solving the homogeneous equation is the more fundamental one.

In this chapter we will concentrate our attention on equations in which the functions P, Q, and R are constants. In this case, Eq. (7) becomes

$$ay'' + by' + cy = 0, \tag{8}$$

where a, b, and c are given constants. It turns out that Eq. (8) can always be solved easily in terms of the elementary functions of calculus. On the other hand, it is usually much more difficult to solve Eq. (7) if the coefficients are not constants, and a treatment of that case is deferred until Chapter 5.

Before taking up Eq. (8), let us first gain some experience by looking at a simple, but typical, example. Consider the equation

$$y'' - y = 0, \tag{9}$$

which is just Eq. (8) with $a = 1$, $b = 0$, and $c = -1$. In words, Eq. (9) says that we seek a function with the property that the second derivative of the function is the same as the function itself. A little thought will probably produce at least one well-known function from calculus with this property, namely, $y_1(t) = e^t$, the exponential function. A little more thought may also produce a second function, $y_2(t) = e^{-t}$. Some further experimentation reveals that constant multiples of these two solutions are also solutions. For example, the functions $2e^t$ and $5e^{-t}$ also satisfy Eq. (9), as you can verify by calculating their second derivatives. In the same way, the functions $c_1y_1(t) = c_1e^t$ and $c_2y_2(t) = c_2e^{-t}$ satisfy the differential equation (9) for all values of the constants c_1 and c_2. Next, it is of paramount importance to notice that any sum of solutions of Eq. (9) is also a solution. In particular, since $c_1y_1(t)$ and $c_2y_2(t)$ are solutions of Eq. (9), so is the function

$$y = c_1y_1(t) + c_2y_2(t) = c_1e^t + c_2e^{-t} \tag{10}$$

for any values of c_1 and c_2. Again, this can be verified by calculating the second derivative y'' from Eq. (10). We have $y' = c_1e^t - c_2e^{-t}$ and $y'' = c_1e^t + c_2e^{-t}$; thus y'' is the same as y, and Eq. (9) is satisfied.

Let us summarize what we have done so far in this example. Once we notice that the functions $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ are solutions of Eq. (9), it follows that the general linear combination (10) of these functions is also a solution. Since the coefficients c_1 and c_2 in Eq. (10) are arbitrary, this expression represents a doubly infinite family of solutions of the differential equation (9).

It is now possible to consider how to pick out a particular member of this infinite family of solutions that also satisfies a given set of initial conditions. For example, suppose that we want the solution of Eq. (9) that also satisfies the initial conditions

$$y(0) = 2, \qquad y'(0) = -1. \tag{11}$$

In other words, we seek the solution that passes through the point (0, 2) and at that point has the slope -1. First, we set $t = 0$ and $y = 2$ in Eq. (10); this gives the equation

$$c_1 + c_2 = 2. \tag{12}$$

Next, we differentiate Eq. (10) with the result that

$$y' = c_1 e^t - c_2 e^{-t}.$$

Then, setting $t = 0$ and $y' = -1$, we obtain

$$c_1 - c_2 = -1. \tag{13}$$

By solving Eqs. (12) and (13) simultaneously for c_1 and c_2 we find that

$$c_1 = \tfrac{1}{2}, \qquad c_2 = \tfrac{3}{2}. \tag{14}$$

Finally, inserting these values in Eq. (10), we obtain

$$y = \tfrac{1}{2}e^t + \tfrac{3}{2}e^{-t}, \tag{15}$$

the solution of the initial value problem consisting of the differential equation (9) and the initial conditions (11).

We now return to the more general equation (8),

$$ay'' + by' + cy = 0,$$

which has arbitrary (real) constant coefficients. Based on our experience with Eq. (9), let us also seek exponential solutions of Eq. (8). Thus we suppose that $y = e^{rt}$, where r is a parameter to be determined. Then it follows that $y' = re^{rt}$ and $y'' = r^2 e^{rt}$. By substituting these expressions for y, y', and y'' in Eq. (8), we obtain

$$(ar^2 + br + c)e^{rt} = 0,$$

or, since $e^{rt} \neq 0$,

$$ar^2 + br + c = 0. \tag{16}$$

Equation (16) is called the **characteristic equation** for the differential equation (8). Its significance lies in the fact that if r is a root of the polynomial equation (16), then $y = e^{rt}$ is a solution of the differential equation (8). Since Eq. (16) is a quadratic equation with real coefficients, it has two roots, which may be real and different, real but repeated, or complex conjugates. We consider the first case here, and the latter two cases in Sections 3.4 and 3.5.

Assuming that the roots of the characteristic equation (16) are real and different, let them be denoted by r_1 and r_2, where $r_1 \neq r_2$. Then $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are two solutions of Eq. (8). Just as in the preceding example, it now follows that

$$y = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{17}$$

is also a solution of Eq. (8). To verify that this is so, we can differentiate the expression in Eq. (17); hence

$$y' = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t} \tag{18}$$

and

$$y'' = c_1 r_1^2 e^{r_1 t} + c_2 r_2^2 e^{r_2 t}. \tag{19}$$

Substituting these expressions for y, y', and y'' in Eq. (8) and rearranging terms, we obtain

$$ay'' + by' + cy = c_1(ar_1^2 + br_1 + c)e^{r_1 t} + c_2(ar_2^2 + br_2 + c)e^{r_2 t}. \tag{20}$$

The quantity in each of the parentheses on the right side of Eq. (20) is zero because r_1 and r_2 are roots of Eq. (16); therefore, y as given by Eq. (17) is indeed a solution of Eq. (8), as we wished to verify.

Now suppose that we want to find the particular member of the family of solutions (17) that satisfies the initial conditions (6),

$$y(t_0) = y_0, \qquad y'(t_0) = y_0'.$$

By substituting $t = t_0$ and $y = y_0$ in Eq. (17), we obtain

$$c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} = y_0. \tag{21}$$

Similarly, setting $t = t_0$ and $y' = y_0'$ in Eq. (18) gives

$$c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} = y_0'. \tag{22}$$

On solving Eqs. (21) and (22) simultaneously for c_1 and c_2, we find that

$$c_1 = \frac{y_0' - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0}, \qquad c_2 = \frac{y_0 r_1 - y_0'}{r_1 - r_2} e^{-r_2 t_0}. \tag{23}$$

Thus, no matter what initial conditions are assigned, that is, regardless of the values of t_0, y_0, and y_0' in Eqs. (6), it is always possible to determine c_1 and c_2 so that the initial conditions are satisfied; moreover, there is only one possible choice of c_1 and c_2 for each set of initial conditions. With the values of c_1 and c_2 given by Eq. (23), the expression (17) is the solution of the initial value problem

$$ay'' + by' + cy = 0, \qquad y(t_0) = y_0, \quad y'(t_0) = y_0'. \tag{24}$$

It is possible to show, on the basis of the fundamental theorem cited in the next section, that all solutions of Eq. (8) are included in the expression (17), at least for the case in which the roots of Eq. (16) are real and different. Therefore, we call Eq. (17) the general solution of Eq. (8). The fact that any possible initial conditions can be satisfied by the proper choice of the constants in Eq. (17) makes more plausible the idea that this expression does include all solutions of Eq. (8).

EXAMPLE 1

Find the general solution of

$$y'' + 5y' + 6y = 0. \tag{25}$$

We assume that $y = e^{rt}$, and it then follows that r must be a root of the characteristic equation

$$r^2 + 5r + 6 = (r + 2)(r + 3) = 0.$$

Thus the possible values of r are $r_1 = -2$ and $r_2 = -3$; the general solution of Eq. (25) is

$$y = c_1 e^{-2t} + c_2 e^{-3t}. \tag{26}$$

EXAMPLE 2

Find the solution of the initial value problem

$$y'' + 5y' + 6y = 0, \qquad y(0) = 2, \quad y'(0) = 3. \tag{27}$$

The general solution of the differential equation was found in Example 1 and is given by Eq. (26). To satisfy the first initial condition we set $t = 0$ and $y = 2$ in Eq. (26); thus c_1 and c_2 must satisfy

$$c_1 + c_2 = 2. \tag{28}$$

To use the second initial condition we must first differentiate Eq. (26). This gives $y' = -2c_1e^{-2t} - 3c_2e^{-3t}$. Then, setting $t = 0$ and $y' = 3$, we obtain

$$-2c_1 - 3c_2 = 3. \tag{29}$$

By solving Eqs. (28) and (29) we find that $c_1 = 9$ and $c_2 = -7$. Using these values in the expression (26), we obtain the solution

$$y = 9e^{-2t} - 7e^{-3t} \tag{30}$$

of the initial value problem (27). The graph of the solution is shown in Figure 3.1.1.

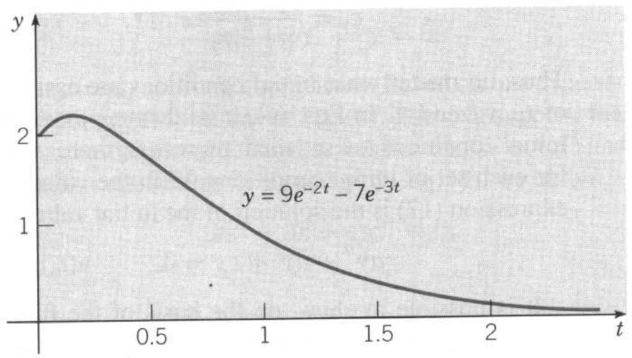

FIGURE 3.1.1 Solution of $y'' + 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 3$.

EXAMPLE 3

Find the solution of the initial value problem

$$4y'' - 8y' + 3y = 0, \qquad y(0) = 2, \quad y'(0) = \tfrac{1}{2}. \tag{31}$$

If $y = e^{rt}$, then the characteristic equation is

$$4r^2 - 8r + 3 = 0$$

and its roots are $r = 3/2$ and $r = 1/2$. Therefore the general solution of the differential equation is

$$y = c_1e^{3t/2} + c_2e^{t/2}. \tag{32}$$

Applying the initial conditions, we obtain the following two equations for c_1 and c_2:

$$c_1 + c_2 = 2, \qquad \tfrac{3}{2}c_1 + \tfrac{1}{2}c_2 = \tfrac{1}{2}.$$

The solution of these equations is $c_1 = -\frac{1}{2}$, $c_2 = \frac{5}{2}$, and the solution of the initial value problem (31) is

$$y = -\tfrac{1}{2}e^{3t/2} + \tfrac{5}{2}e^{t/2}. \tag{33}$$

Figure 3.1.2 shows the graph of the solution.

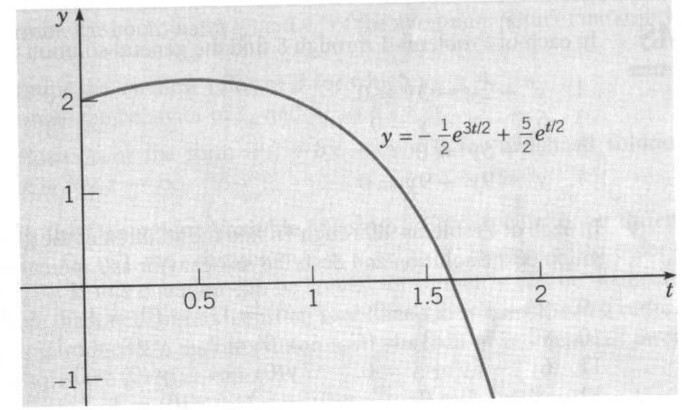

FIGURE 3.1.2 Solution of $4y'' - 8y' + 3y = 0$, $y(0) = 2$, $y'(0) = 0.5$.

EXAMPLE 4

The solution (30) of the initial value problem (27) initially increases (because its initial slope is positive) but eventually approaches zero (because both terms involve negative exponential functions). Therefore the solution must have a maximum point and the graph in Figure 3.1.1 confirms this. Determine the location of this maximum point.

One can estimate the coordinates of the maximum point from the graph, but to find them more precisely we seek the point where the solution has a horizontal tangent line. By differentiating the solution (30), $y = 9e^{-2t} - 7e^{-3t}$, with respect to t we obtain

$$y' = -18e^{-2t} + 21e^{-3t}. \tag{34}$$

Setting y' equal to zero and multiplying by e^{3t}, we find that the critical value t_c satisfies $e^t = 7/6$; hence

$$t_c = \ln(7/6) \cong 0.15415. \tag{35}$$

The corresponding maximum value y_M is given by

$$y_M = 9e^{-2t_c} - 7e^{-3t_c} = \frac{108}{49} \cong 2.20408. \tag{36}$$

In this example the initial slope is 3, but the solution of the given differential equation behaves in a similar way for any other positive initial slope. In Problem 26 you are asked to determine how the coordinates of the maximum point depend on the initial slope.

Returning to the equation $ay'' + by' + cy = 0$ with arbitrary coefficients, recall that when $r_1 \neq r_2$, its general solution (17) is the sum of two exponential functions. Therefore the solution has a relatively simple geometrical behavior: as t increases, the magnitude of the solution either tends to zero (when both exponents are negative) or else grows rapidly (when at least one exponent is positive). These two cases are illustrated by the solutions of Examples 2 and 3, which are shown in Figures 3.1.1 and 3.1.2, respectively. There is also a third case that occurs less often; the solution approaches a constant when one exponent is zero and the other is negative.

PROBLEMS

In each of Problems 1 through 8 find the general solution of the given differential equation.

1. $y'' + 2y' - 3y = 0$
2. $y'' + 3y' + 2y = 0$
3. $6y'' - y' - y = 0$
4. $2y'' - 3y' + y = 0$
5. $y'' + 5y' = 0$
6. $4y'' - 9y = 0$
7. $y'' - 9y' + 9y = 0$
8. $y'' - 2y' - 2y = 0$

In each of Problems 9 through 16 find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as t increases.

9. $y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 1$
10. $y'' + 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -1$
11. $6y'' - 5y' + y = 0, \quad y(0) = 4, \quad y'(0) = 0$
12. $y'' + 3y' = 0, \quad y(0) = -2, \quad y'(0) = 3$
13. $y'' + 5y' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 0$
14. $2y'' + y' - 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$
15. $y'' + 8y' - 9y = 0, \quad y(1) = 1, \quad y'(1) = 0$
16. $4y'' - y = 0, \quad y(-2) = 1, \quad y'(-2) = -1$
17. Find a differential equation whose general solution is $y = c_1e^{2t} + c_2e^{-3t}$.
18. Find a differential equation whose general solution is $y = c_1e^{-t/2} + c_2e^{-2t}$.
19. ▶ Find the solution of the initial value problem

$$y'' - y = 0, \qquad y(0) = \tfrac{5}{4}, \quad y'(0) = -\tfrac{3}{4}.$$

Plot the solution for $0 \le t \le 2$ and determine its minimum value.

20. Find the solution of the initial value problem

$$2y'' - 3y' + y = 0, \qquad y(0) = 2, \quad y'(0) = \tfrac{1}{2}.$$

Then determine the maximum value of the solution and also find the point where the solution is zero.

21. Solve the initial value problem $y'' - y' - 2y = 0$, $y(0) = \alpha$, $y'(0) = 2$. Then find α so that the solution approaches zero as $t \to \infty$.
22. Solve the initial value problem $4y'' - y = 0$, $y(0) = 2$, $y'(0) = \beta$. Then find β so that the solution approaches zero as $t \to \infty$.

In each of Problems 23 and 24 determine the values of α, if any, for which all solutions tend to zero as $t \to \infty$; also determine the values of α, if any, for which all (nonzero) solutions become unbounded as $t \to \infty$.

23. $y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0$
24. $y'' + (3 - \alpha)y' - 2(\alpha - 1)y = 0$
25. ▶ Consider the initial value problem

$$2y'' + 3y' - 2y = 0, \qquad y(0) = 1, \quad y'(0) = -\beta,$$

where $\beta > 0$.
(a) Solve the initial value problem.
(b) Plot the solution when $\beta = 1$. Find the coordinates (t_0, y_0) of the minimum point of the solution in this case.
(c) Find the smallest value of β for which the solution has no minimum point.

26. ▶ Consider the initial value problem (see Example 4)

$$y'' + 5y' + 6y = 0, \qquad y(0) = 2, \quad y'(0) = \beta,$$

where $\beta > 0$.
(a) Solve the initial value problem.

(b) Determine the coordinates t_m and y_m of the maximum point of the solution as functions of β.
(c) Determine the smallest value of β for which $y_m \ge 4$.
(d) Determine the behavior of t_m and y_m as $\beta \to \infty$.

27. Find an equation of the form $ay'' + by' + cy = 0$ for which all solutions approach a multiple of e^{-t} as $t \to \infty$.

Equations with the Dependent Variable Missing. For a second order differential equation of the form $y'' = f(t, y')$, the substitution $v = y'$, $v' = y''$ leads to a first order equation of the form $v' = f(t, v)$. If this equation can be solved for v, then y can be obtained by integrating $dy/dt = v$. Note that one arbitrary constant is obtained in solving the first order equation for v, and a second is introduced in the integration for y. In each of Problems 28 through 33 use this substitution to solve the given equation.

28. $t^2y'' + 2ty' - 1 = 0, \quad t > 0$
29. $ty'' + y' = 1, \quad t > 0$
30. $y'' + t(y')^2 = 0$
31. $2t^2y'' + (y')^3 = 2ty', \quad t > 0$
32. $y'' + y' = e^{-t}$
33. $t^2y'' = (y')^2, \quad t > 0$

Equations with the Independent Variable Missing. If a second order differential equation has the form $y'' = f(y, y')$, then the independent variable t does not appear explicitly, but only through the dependent variable y. If we let $v = y'$, then we obtain $dv/dt = f(y, v)$. Since the right side of this equation depends on y and v, rather than on t and v, this equation is not of the form of the first order equations discussed in Chapter 2. However, if we think of y as the independent variable, then by the chain rule $dv/dt = (dv/dy)(dy/dt) = v(dv/dy)$. Hence the original differential equation can be written as $v(dv/dy) = f(y, v)$. Provided that this first order equation can be solved, we obtain v as a function of y. A relation between y and t results from solving $dy/dt = v(y)$. Again, there are two arbitrary constants in the final result. In each of Problems 34 through 39 use this method to solve the given differential equation.

34. $yy'' + (y')^2 = 0$
35. $y'' + y = 0$
36. $y'' + y(y')^3 = 0$
37. $2y^2y'' + 2y(y')^2 = 1$
38. $yy'' - (y')^3 = 0$
39. $y'' + (y')^2 = 2e^{-y}$

Hint: In Problem 39 the transformed equation is a Bernoulli equation. See Problem 27 in Section 2.4.

In each of Problems 40 through 43 solve the given initial value problem using the methods of Problems 28 through 39.

40. $y'y'' = 2, \quad y(0) = 1, \quad y'(0) = 2$
41. $y'' - 3y^2 = 0, \quad y(0) = 2, \quad y'(0) = 4$
42. $(1 + t^2)y'' + 2ty' + 3t^{-2} = 0, \quad y(1) = 2, \quad y'(1) = -1$
43. $y'y'' - t = 0, \quad y(1) = 2, \quad y'(1) = 1$

3.2 Fundamental Solutions of Linear Homogeneous Equations

In the preceding section we showed how to solve some differential equations of the form

$$ay'' + by' + cy = 0,$$

where a, b, and c are constants. Now we build on those results to provide a clearer picture of the structure of the solutions of all second order linear homogeneous equations. In

PROBLEMS

In each of Problems 1 through 8 find the general solution of the given differential equation.

1. $y'' + 2y' - 3y = 0$
2. $y'' + 3y' + 2y = 0$
3. $6y'' - y' - y = 0$
4. $2y'' - 3y' + y = 0$
5. $y'' + 5y' = 0$
6. $4y'' - 9y = 0$
7. $y'' - 9y' + 9y = 0$
8. $y'' - 2y' - 2y = 0$

In each of Problems 9 through 16 find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as t increases.

9. $y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 1$
10. $y'' + 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -1$
11. $6y'' - 5y' + y = 0, \quad y(0) = 4, \quad y'(0) = 0$
12. $y'' + 3y' = 0, \quad y(0) = -2, \quad y'(0) = 3$
13. $y'' + 5y' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 0$
14. $2y'' + y' - 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$
15. $y'' + 8y' - 9y = 0, \quad y(1) = 1, \quad y'(1) = 0$
16. $4y'' - y = 0, \quad y(-2) = 1, \quad y'(-2) = -1$
17. Find a differential equation whose general solution is $y = c_1 e^{2t} + c_2 e^{-3t}$.
18. Find a differential equation whose general solution is $y = c_1 e^{-t/2} + c_2 e^{-2t}$.
19. ▶ Find the solution of the initial value problem

$$y'' - y = 0, \qquad y(0) = \tfrac{5}{4}, \quad y'(0) = -\tfrac{3}{4}.$$

Plot the solution for $0 \le t \le 2$ and determine its minimum value.

20. Find the solution of the initial value problem

$$2y'' - 3y' + y = 0, \qquad y(0) = 2, \quad y'(0) = \tfrac{1}{2}.$$

Then determine the maximum value of the solution and also find the point where the solution is zero.

21. Solve the initial value problem $y'' - y' - 2y = 0$, $y(0) = \alpha$, $y'(0) = 2$. Then find α so that the solution approaches zero as $t \to \infty$.
22. Solve the initial value problem $4y'' - y = 0$, $y(0) = 2$, $y'(0) = \beta$. Then find β so that the solution approaches zero as $t \to \infty$.

In each of Problems 23 and 24 determine the values of α, if any, for which all solutions tend to zero as $t \to \infty$; also determine the values of α, if any, for which all (nonzero) solutions become unbounded as $t \to \infty$.

23. $y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0$
24. $y'' + (3 - \alpha)y' - 2(\alpha - 1)y = 0$
25. ▶ Consider the initial value problem

$$2y'' + 3y' - 2y = 0, \qquad y(0) = 1, \quad y'(0) = -\beta,$$

where $\beta > 0$.
(a) Solve the initial value problem.
(b) Plot the solution when $\beta = 1$. Find the coordinates (t_0, y_0) of the minimum point of the solution in this case.
(c) Find the smallest value of β for which the solution has no minimum point.

26. ▶ Consider the initial value problem (see Example 4)

$$y'' + 5y' + 6y = 0, \qquad y(0) = 2, \quad y'(0) = \beta,$$

where $\beta > 0$.
(a) Solve the initial value problem.
(b) Determine the coordinates t_m and y_m of the maximum point of the solution as functions of β.
(c) Determine the smallest value of β for which $y_m \ge 4$.
(d) Determine the behavior of t_m and y_m as $\beta \to \infty$.

27. Find an equation of the form $ay'' + by' + cy = 0$ for which all solutions approach a multiple of e^{-t} as $t \to \infty$.

Equations with the Dependent Variable Missing. For a second order differential equation of the form $y'' = f(t, y')$, the substitution $v = y'$, $v' = y''$ leads to a first order equation of the form $v' = f(t, v)$. If this equation can be solved for v, then y can be obtained by integrating $dy/dt = v$. Note that one arbitrary constant is obtained in solving the first order equation for v, and a second is introduced in the integration for y. In each of Problems 28 through 33 use this substitution to solve the given equation.

28. $t^2 y'' + 2ty' - 1 = 0, \quad t > 0$
29. $ty'' + y' = 1, \quad t > 0$
30. $y'' + t(y')^2 = 0$
31. $2t^2 y'' + (y')^3 = 2ty', \quad t > 0$
32. $y'' + y' = e^{-t}$
33. $t^2 y'' = (y')^2, \quad t > 0$

Equations with the Independent Variable Missing. If a second order differential equation has the form $y'' = f(y, y')$, then the independent variable t does not appear explicitly, but only through the dependent variable y. If we let $v = y'$, then we obtain $dv/dt = f(y, v)$. Since the right side of this equation depends on y and v, rather than on t and v, this equation is not of the form of the first order equations discussed in Chapter 2. However, if we think of y as the independent variable, then by the chain rule $dv/dt = (dv/dy)(dy/dt) = v(dv/dy)$. Hence the original differential equation can be written as $v(dv/dy) = f(y, v)$. Provided that this first order equation can be solved, we obtain v as a function of y. A relation between y and t results from solving $dy/dt = v(y)$. Again, there are two arbitrary constants in the final result. In each of Problems 34 through 39 use this method to solve the given differential equation.

34. $yy'' + (y')^2 = 0$
35. $y'' + y = 0$
36. $y'' + y(y')^3 = 0$
37. $2y^2 y'' + 2y(y')^2 = 1$
38. $yy'' - (y')^3 = 0$
39. $y'' + (y')^2 = 2e^{-y}$

Hint: In Problem 39 the transformed equation is a Bernoulli equation. See Problem 27 in Section 2.4.

In each of Problems 40 through 43 solve the given initial value problem using the methods of Problems 28 through 39.

40. $y'y'' = 2, \quad y(0) = 1, \quad y'(0) = 2$
41. $y'' - 3y^2 = 0, \quad y(0) = 2, \quad y'(0) = 4$
42. $(1 + t^2)y'' + 2ty' + 3t^{-2} = 0, \quad y(1) = 2, \quad y'(1) = -1$
43. $y'y'' - t = 0, \quad y(1) = 2, \quad y'(1) = 1$

3.2 Fundamental Solutions of Linear Homogeneous Equations

In the preceding section we showed how to solve some differential equations of the form

$$ay'' + by' + cy = 0,$$

where a, b, and c are constants. Now we build on those results to provide a clearer picture of the structure of the solutions of all second order linear homogeneous equations. In

nth order linear homogeneous differential equation forms a vector space of dimension n, and that any set of n linearly independent solutions of the differential equation forms a basis for the space. This connection between differential equations and vectors constitutes a good reason for the study of abstract linear algebra.

PROBLEMS

In each of Problems 1 through 8 determine whether the given pair of functions is linearly independent or linearly dependent.

1. $f(t) = t^2 + 5t, \quad g(t) = t^2 - 5t$
2. $f(\theta) = \cos 3\theta, \quad g(\theta) = 4\cos^3\theta - 3\cos\theta$
3. $f(t) = e^{\lambda t}\cos\mu t, \quad g(t) = e^{\lambda t}\sin\mu t, \quad \mu \neq 0$
4. $f(x) = e^{3x}, \quad g(x) = e^{3(x-1)}$
5. $f(t) = 3t - 5, \quad g(t) = 9t - 15$
6. $f(t) = t, \quad g(t) = t^{-1}$
7. $f(t) = 3t, \quad g(t) = |t|$
8. $f(x) = x^3, \quad g(x) = |x|^3$
9. The Wronskian of two functions is $W(t) = t\sin^2 t$. Are the functions linearly independent or linearly dependent? Why?
10. The Wronskian of two functions is $W(t) = t^2 - 4$. Are the functions linearly independent or linearly dependent? Why?
11. If the functions y_1 and y_2 are linearly independent solutions of $y'' + p(t)y' + q(t)y = 0$, prove that c_1y_1 and c_2y_2 are also linearly independent solutions, provided that neither c_1 nor c_2 is zero.
12. If the functions y_1 and y_2 are linearly independent solutions of $y'' + p(t)y' + q(t)y = 0$, prove that $y_3 = y_1 + y_2$ and $y_4 = y_1 - y_2$ also form a linearly independent set of solutions. Conversely, if y_3 and y_4 are linearly independent solutions of the differential equation, show that y_1 and y_2 are also.
13. If the functions y_1 and y_2 are linearly independent solutions of $y'' + p(t)y' + q(t)y = 0$, determine under what conditions the functions $y_3 = a_1y_1 + a_2y_2$ and $y_4 = b_1y_1 + b_2y_2$ also form a linearly independent set of solutions.
14. (a) Prove that any two-dimensional vector can be written as a linear combination of $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} - \mathbf{j}$.
(b) Prove that if the vectors $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j}$ and $\mathbf{y} = y_1\mathbf{i} + y_2\mathbf{j}$ are linearly independent, then any vector $\mathbf{z} = z_1\mathbf{i} + z_2\mathbf{j}$ can be expressed as a linear combination of $\mathbf{x}$ and $\mathbf{y}$. Note that if $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, then $x_1y_2 - x_2y_1 \neq 0$. Why?

In each of Problems 15 through 18 find the Wronskian of two solutions of the given differential equation without solving the equation.

15. $t^2y'' - t(t+2)y' + (t+2)y = 0$
16. $(\cos t)y'' + (\sin t)y' - ty = 0$
17. $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, Bessel's equation
18. $(1 - x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$, Legendre's equation
19. Show that if p is differentiable and $p(t) > 0$, then the Wronskian $W(t)$ of two solutions of $[p(t)y']' + q(t)y = 0$ is $W(t) = c/p(t)$, where c is a constant.
20. If y_1 and y_2 are linearly independent solutions of $ty'' + 2y' + te^ty = 0$ and if $W(y_1, y_2)(1) = 2$, find the value of $W(y_1, y_2)(5)$.
21. If y_1 and y_2 are linearly independent solutions of $t^2y'' - 2y' + (3+t)y = 0$ and if $W(y_1, y_2)(2) = 3$, find the value of $W(y_1, y_2)(4)$.
22. If the Wronskian of any two solutions of $y'' + p(t)y' + q(t)y = 0$ is constant, what does this imply about the coefficients p and q?
23. If f, g, and h are differentiable functions, show that $W(fg, fh) = f^2W(g, h)$.

In Problems 24 through 26 assume that p and q are continuous, and that the functions y_1 and y_2 are solutions of the differential equation $y'' + p(t)y' + q(t)y = 0$ on an open interval I.

24. Prove that if y_1 and y_2 are zero at the same point in I, then they cannot be a fundamental set of solutions on that interval.
25. Prove that if y_1 and y_2 have maxima or minima at the same point in I, then they cannot be a fundamental set of solutions on that interval.
26. Prove that if y_1 and y_2 have a common point of inflection t_0 in I, then they cannot be a fundamental set of solutions on I unless both p and q are zero at t_0.
27. Show that t and t^2 are linearly independent on $-1 < t < 1$; indeed, they are linearly independent on every interval. Show also that $W(t, t^2)$ is zero at $t = 0$. What can you conclude from this about the possibility that t and t^2 are solutions of a differential equation $y'' + p(t)y' + q(t)y = 0$? Verify that t and t^2 are solutions of the equation $t^2y'' - 2ty' + 2y = 0$. Does this contradict your conclusion? Does the behavior of the Wronskian of t and t^2 contradict Theorem 3.3.2?
28. Show that the functions $f(t) = t^2|t|$ and $g(t) = t^3$ are linearly dependent on $0 < t < 1$ and on $-1 < t < 0$, but are linearly independent on $-1 < t < 1$. Although f and g are linearly independent there, show that $W(f, g)$ is zero for all t in $-1 < t < 1$. Hence f and g cannot be solutions of an equation $y'' + p(t)y' + q(t)y = 0$ with p and q continuous on $-1 < t < 1$.

3.4 Complex Roots of the Characteristic Equation

We continue our discussion of the equation

$$ay'' + by' + cy = 0, \tag{1}$$

where a, b, and c are given real numbers. In Section 3.1 we found that if we seek solutions of the form $y = e^{rt}$, then r must be a root of the characteristic equation

$$ar^2 + br + c = 0. \tag{2}$$

If the roots r_1 and r_2 are real and different, which occurs whenever the discriminant $b^2 - 4ac$ is positive, then the general solution of Eq. (1) is

$$y = c_1e^{r_1t} + c_2e^{r_2t}. \tag{3}$$

Suppose now that $b^2 - 4ac$ is negative. Then the roots of Eq. (2) are conjugate complex numbers; we denote them by

$$r_1 = \lambda + i\mu, \qquad r_2 = \lambda - i\mu, \tag{4}$$

where λ and μ are real. The corresponding expressions for y are

$$y_1(t) = \exp[(\lambda + i\mu)t], \qquad y_2(t) = \exp[(\lambda - i\mu)t]. \tag{5}$$

Our first task is to explore what is meant by these expressions, which involve evaluating the exponential function for a complex exponent. For example, if $\lambda = -1$, $\mu = 2$, and $t = 3$, then from Eq. (5)

$$y_1(3) = e^{-3+6i}. \tag{6}$$

What does it mean to raise the number e to a complex power? The answer is provided by an important relation known as Euler's formula.

Euler's Formula. To assign a meaning to the expressions in Eqs. (5) we need to give a definition of the complex exponential function. Of course, we want the definition to reduce to the familiar real exponential function when the exponent is real. There are several ways to accomplish this extension of the exponential function. Here we use a method based on infinite series; an alternative is outlined in Problem 28.

Recall from calculus that the Taylor series for e^t about $t = 0$ is

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \qquad -\infty < t < \infty. \tag{7}$$

If we now assume that we can substitute it for t in Eq. (7), then we have

$$\begin{aligned} e^{it} &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!}, \end{aligned} \tag{8}$$

where we have separated the sum into its real and imaginary parts, making use of the fact that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and so forth. The first series in Eq. (8) is precisely the Taylor series for $\cos t$ about $t = 0$, and the second is the Taylor series for $\sin t$ about $t = 0$. Thus we have

$$e^{it} = \cos t + i \sin t. \tag{9}$$

Equation (9) is known as Euler's formula and is an extremely important mathematical relationship. While our derivation of Eq. (9) is based on the unverified assumption that the series (7) can be used for complex as well as real values of the independent variable, our intention is to use this derivation only to make Eq. (9) seem plausible. We now put matters on a firm foundation by adopting Eq. (9) as the *definition* of e^{it}. In other words, whenever we write e^{it}, we mean the expression on the right side of Eq. (9).

There are some variations of Euler's formula that are also worth noting. If we replace t by $-t$ in Eq. (9) and recall that $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$, then we have

$$e^{-it} = \cos t - i \sin t. \tag{10}$$

Further, if t is replaced by μt in Eq. (9), then we obtain a generalized version of Euler's formula, namely,

$$e^{i\mu t} = \cos \mu t + i \sin \mu t. \tag{11}$$

Next, we want to extend the definition of the exponential function to arbitrary complex exponents of the form $(\lambda + i\mu)t$. Since we want the usual properties of the exponential function to hold for complex exponents, we certainly want $\exp[(\lambda + i\mu)t]$ to satisfy

$$e^{(\lambda+i\mu)t} = e^{\lambda t} e^{i\mu t}. \tag{12}$$

Then, substituting for $e^{i\mu t}$ from Eq. (11), we obtain

$$\begin{aligned} e^{(\lambda+i\mu)t} &= e^{\lambda t}(\cos \mu t + i \sin \mu t) \\ &= e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t. \end{aligned} \tag{13}$$

We now take Eq. (13) as the definition of $\exp[(\lambda + i\mu)t]$. The value of the exponential function with a complex exponent is a complex number whose real and imaginary parts are given by the terms on the right side of Eq. (13). Observe that the real and imaginary parts of $\exp[(\lambda + i\mu)t]$ are expressed entirely in terms of elementary real-valued functions. For example, the quantity in Eq. (6) has the value

$$e^{-3+6i} = e^{-3} \cos 6 + i e^{-3} \sin 6 \cong 0.0478041 - 0.0139113i.$$

With the definitions (9) and (13) it is straightforward to show that the usual laws of exponents are valid for the complex exponential function. It is also easy to verify that the differentiation formula

$$\frac{d}{dt}(e^{rt}) = re^{rt} \tag{14}$$

also holds for complex values of r.

Real-Valued Solutions. The functions $y_1(t)$ and $y_2(t)$, given by Eqs. (5) and with the meaning expressed by Eq. (13), are solutions of Eq. (1) when the roots of the characteristic equation (2) are complex numbers $\lambda \pm i\mu$. Unfortunately, the solutions y_1 and y_2 are complex-valued functions, whereas in general we would prefer to have real-valued solutions, if possible, because the differential equation itself has real coefficients. Such solutions can be found as a consequence of Theorem 3.2.2, which states that if y_1 and y_2 are solutions of Eq. (1), then any linear combination of y_1 and y_2 is also a solution. In particular, let us form the sum and then the difference of y_1 and y_2. We have

$$\begin{aligned} y_1(t) + y_2(t) &= e^{\lambda t}(\cos \mu t + i \sin \mu t) + e^{\lambda t}(\cos \mu t - i \sin \mu t) \\ &= 2e^{\lambda t} \cos \mu t \end{aligned}$$

and

$$\begin{aligned} y_1(t) - y_2(t) &= e^{\lambda t}(\cos \mu t + i \sin \mu t) - e^{\lambda t}(\cos \mu t - i \sin \mu t) \\ &= 2ie^{\lambda t} \sin \mu t. \end{aligned}$$

Hence, neglecting the constant multipliers 2 and $2i$, respectively, we have obtained a pair of real-valued solutions

$$u(t) = e^{\lambda t} \cos \mu t, \qquad v(t) = e^{\lambda t} \sin \mu t. \tag{15}$$

Observe that u and v are simply the real and imaginary parts, respectively, of y_1.

By direct computation you can show that the Wronskian of u and v is

$$W(u, v)(t) = \mu e^{2\lambda t}. \tag{16}$$

Thus, as long as $\mu \neq 0$, the Wronskian W is not zero, so u and v form a fundamental set of solutions. (Of course, if $\mu = 0$, then the roots are real and the discussion in this section is not applicable.) Consequently, if the roots of the characteristic equation are complex numbers $\lambda \pm i\mu$, with $\mu \neq 0$, then the general solution of Eq. (1) is

$$y = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t, \tag{17}$$

where c_1 and c_2 are arbitrary constants. Note that the solution (17) can be written down as soon as the values of λ and μ are known.

EXAMPLE 1

Find the general solution of

$$y'' + y' + y = 0. \tag{18}$$

The characteristic equation is

$$r^2 + r + 1 = 0,$$

and its roots are

$$r = \frac{-1 \pm (1-4)^{1/2}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

Thus $\lambda = -1/2$ and $\mu = \sqrt{3}/2$, so the general solution of Eq. (18) is

$$y = c_1 e^{-t/2}\cos(\sqrt{3}t/2) + c_2 e^{-t/2}\sin(\sqrt{3}t/2). \tag{19}$$

EXAMPLE 2

Find the general solution of

$$y'' + 9y = 0. \tag{20}$$

The characteristic equation is $r^2 + 9 = 0$ with the roots $r = \pm 3i$; thus $\lambda = 0$ and $\mu = 3$. The general solution is

$$y = c_1 \cos 3t + c_2 \sin 3t; \tag{21}$$

note that if the real part of the roots is zero, as in this example, then there is no exponential factor in the solution.

EXAMPLE 3

Find the solution of the initial value problem

$$16y'' - 8y' + 145y = 0, \qquad y(0) = -2, \quad y'(0) = 1. \tag{22}$$

The characteristic equation is $16r^2 - 8r + 145 = 0$ and its roots are $r = 1/4 \pm 3i$. Thus the general solution of the differential equation is

$$y = c_1 e^{t/4}\cos 3t + c_2 e^{t/4}\sin 3t. \tag{23}$$

To apply the first initial condition we set $t = 0$ in Eq. (23); this gives

$$y(0) = c_1 = -2.$$

For the second initial condition we must differentiate Eq. (23) and then set $t = 0$. In this way we find that

$$y'(0) = \tfrac{1}{4}c_1 + 3c_2 = 1,$$

from which $c_2 = 1/2$. Using these values of c_1 and c_2 in Eq. (23), we obtain

$$y = -2e^{t/4}\cos 3t + \tfrac{1}{2}e^{t/4}\sin 3t \tag{24}$$

as the solution of the initial value problem (22).

We will discuss the geometrical properties of solutions such as these more fully in Section 3.8, so we will be very brief here. Each of the solutions u and v in Eqs. (15) represents an oscillation, because of the trigonometric factors, and also either grows or

decays exponentially, depending on the sign of λ (unless $\lambda = 0$). In Example 1 we have $\lambda = -1/2 < 0$, so solutions are decaying oscillations. The graph of a typical solution of Eq. (18) is shown in Figure 3.4.1. On the other hand, $\lambda = 1/4 > 0$ in Example 3, so solutions of the differential equation (22) are growing oscillations. The graph of the solution (24) of the given initial value problem is shown in Figure 3.4.2. The intermediate case is illustrated in Example 2 in which $\lambda = 0$. In this case the solution neither grows nor decays exponentially, but oscillates steadily; a typical solution of Eq. (20) is shown in Figure 3.4.3.

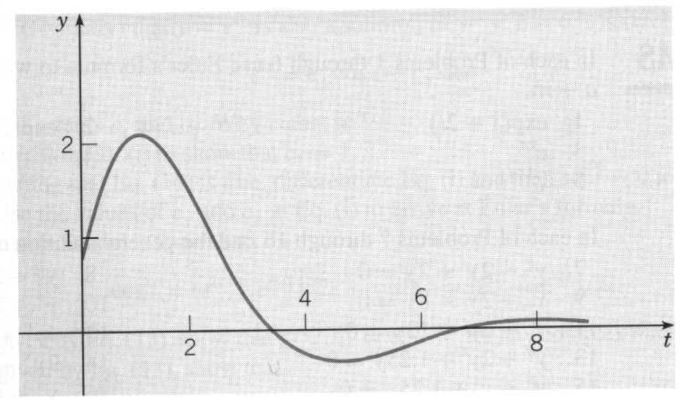

FIGURE 3.4.1 A typical solution of $y'' + y' + y = 0$.

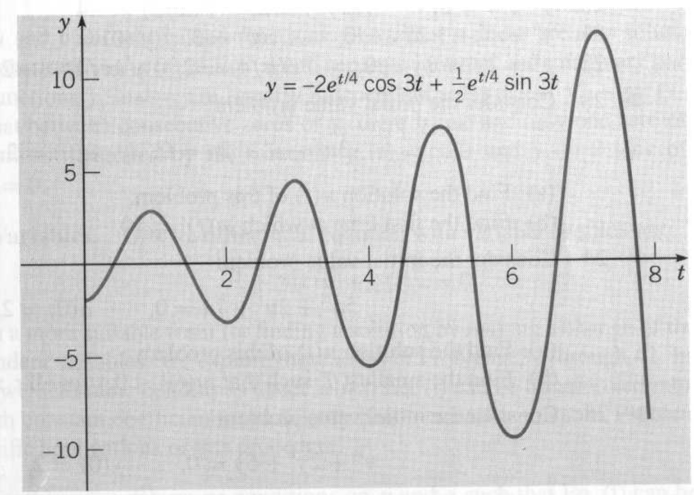

FIGURE 3.4.2 Solution of $16y'' - 8y' + 145y = 0$, $y(0) = -2$, $y'(0) = 1$.

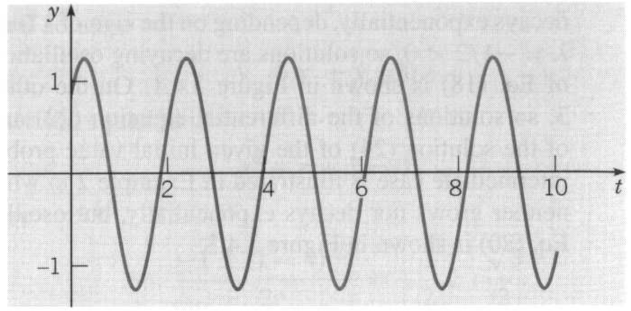

FIGURE 3.4.3 A typical solution of $y'' + 9y = 0$.

PROBLEMS

In each of Problems 1 through 6 use Euler's formula to write the given expression in the form $a + ib$.

1. $\exp(1 + 2i)$
2. $\exp(2 - 3i)$
3. $e^{i\pi}$
4. $e^{2-(\pi/2)i}$
5. 2^{1-i}
6. π^{-1+2i}

In each of Problems 7 through 16 find the general solution of the given differential equation.

7. $y'' - 2y' + 2y = 0$
8. $y'' - 2y' + 6y = 0$
9. $y'' + 2y' - 8y = 0$
10. $y'' + 2y' + 2y = 0$
11. $y'' + 6y' + 13y = 0$
12. $4y'' + 9y = 0$
13. $y'' + 2y' + 1.25y = 0$
14. $9y'' + 9y' - 4y = 0$
15. $y'' + y' + 1.25y = 0$
16. $y'' + 4y' + 6.25y = 0$

In each of Problems 17 through 22 find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

17. $y'' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$
18. $y'' + 4y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 0$
19. $y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2$
20. $y'' + y = 0, \quad y(\pi/3) = 2, \quad y'(\pi/3) = -4$
21. $y'' + y' + 1.25y = 0, \quad y(0) = 3, \quad y'(0) = 1$
22. $y'' + 2y' + 2y = 0, \quad y(\pi/4) = 2, \quad y'(\pi/4) = -2$

▶ 23. Consider the initial value problem

$$3u'' - u' + 2u = 0, \qquad u(0) = 2, \quad u'(0) = 0.$$

(a) Find the solution $u(t)$ of this problem.
(b) Find the first time at which $|u(t)| = 10$.

▶ 24. Consider the initial value problem

$$5u'' + 2u' + 7u = 0, \qquad u(0) = 2, \quad u'(0) = 1.$$

(a) Find the solution $u(t)$ of this problem.
(b) Find the smallest T such that $|u(t)| \leq 0.1$ for all $t > T$.

▶ 25. Consider the initial value problem

$$y'' + 2y' + 6y = 0, \qquad y(0) = 2, \quad y'(0) = \alpha \geq 0.$$

(a) Find the solution $y(t)$ of this problem.
(b) Find α so that $y = 0$ when $t = 1$.

(c) Find, as a function of α, the smallest positive value of t for which $y = 0$.
(d) Determine the limit of the expression found in part (c) as $\alpha \to \infty$.

▶ 26. Consider the initial value problem

$$y'' + 2ay' + (a^2 + 1)y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

(a) Find the solution $y(t)$ of this problem.
(b) For $a = 1$ find the smallest T such that $|y(t)| < 0.1$ for $t > T$.
(c) Repeat part (b) for $a = 1/4$, $1/2$, and 2.
(d) Using the results of parts (b) and (c), plot T versus a and describe the relation between T and a.

27. Show that $W(e^{\lambda t}\cos\mu t, e^{\lambda t}\sin\mu t) = \mu e^{2\lambda t}$.
28. In this problem we outline a different derivation of Euler's formula.
(a) Show that $y_1(t) = \cos t$ and $y_2(t) = \sin t$ are a fundamental set of solutions of $y'' + y = 0$; that is, show that they are solutions and that their Wronskian is not zero.
(b) Show (formally) that $y = e^{it}$ is also a solution of $y'' + y = 0$. Therefore,

$$e^{it} = c_1 \cos t + c_2 \sin t \qquad \text{(i)}$$

for some constants c_1 and c_2. Why is this so?
(c) Set $t = 0$ in Eq. (i) to show that $c_1 = 1$.
(d) Assuming that Eq. (14) is true, differentiate Eq. (i) and then set $t = 0$ to conclude that $c_2 = i$. Use the values of c_1 and c_2 in Eq. (i) to arrive at Euler's formula.

29. Using Euler's formula, show that

$$\cos t = (e^{it} + e^{-it})/2, \qquad \sin t = (e^{it} - e^{-it})/2i.$$

30. If e^{rt} is given by Eq. (13), show that $e^{(r_1+r_2)t} = e^{r_1 t}e^{r_2 t}$ for any complex numbers r_1 and r_2.
31. If e^{rt} is given by Eq. (13), show that

$$\frac{d}{dt}e^{rt} = re^{rt}$$

for any complex number r.

32. Let the real-valued functions p and q be continuous on the open interval I, and let $y = \phi(t) = u(t) + iv(t)$ be a complex-valued solution of

$$y'' + p(t)y' + q(t)y = 0, \qquad \text{(i)}$$

where u and v are real-valued functions. Show that u and v are also solutions of Eq. (i).
Hint: Substitute $y = \phi(t)$ in Eq. (i) and separate into real and imaginary parts.

33. If the functions y_1 and y_2 are linearly independent solutions of $y'' + p(t)y' + q(t)y = 0$, show that between consecutive zeros of y_1 there is one and only one zero of y_2. Note that this result is illustrated by the solutions $y_1(t) = \cos t$ and $y_2(t) = \sin t$ of the equation $y'' + y = 0$.

Change of Variables. Often a differential equation with variable coefficients,

$$y'' + p(t)y' + q(t)y = 0, \qquad \text{(i)}$$

can be put in a more suitable form for finding a solution by making a change of the independent and/or dependent variables. We explore these ideas in Problems 34 through 42. In particular, in Problem 34 we determine conditions under which Eq. (i) can be transformed into a differential equation with constant coefficients and thereby becomes easily solvable. Problems 35 through 42 give specific applications of this procedure.

34. In this problem we determine conditions on p and q such that Eq. (i) can be transformed into an equation with constant coefficients by a change of the independent variable. Let

$x = u(t)$ be the new independent variable, with the relation between x and t to be specified later.

(a) Show that

$$\frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx}, \qquad \frac{d^2y}{dt^2} = \left(\frac{dx}{dt}\right)^2 \frac{d^2y}{dx^2} + \frac{d^2x}{dt^2}\frac{dy}{dx}.$$

(b) Show that the differential equation (i) becomes

$$\left(\frac{dx}{dt}\right)^2 \frac{d^2y}{dx^2} + \left(\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right)\frac{dy}{dx} + q(t)y = 0. \qquad \text{(ii)}$$

(c) In order for Eq. (ii) to have constant coefficients, the coefficients of d^2y/dx^2 and of y must be proportional. If $q(t) > 0$, then we can choose the constant of proportionality to be 1; hence

$$x = u(t) = \int [q(t)]^{1/2}\, dt. \qquad \text{(iii)}$$

(d) With x chosen as in part (c) show that the coefficient of dy/dx in Eq. (ii) is also a constant, provided that the expression

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} \qquad \text{(iv)}$$

is a constant. Thus Eq. (i) can be transformed into an equation with constant coefficients by a change of the independent variable, provided that the function $(q' + 2pq)/q^{3/2}$ is a constant. How must this result be modified if $q(t) < 0$?

In each of Problems 35 through 37 try to transform the given equation into one with constant coefficients by the method of Problem 34. If this is possible, find the general solution of the given equation.

35. $y'' + ty' + e^{-t^2}y = 0, \qquad -\infty < t < \infty$
36. $y'' + 3ty' + t^2y = 0, \qquad -\infty < t < \infty$
37. $ty'' + (t^2 - 1)y' + t^3y = 0, \qquad 0 < t < \infty$
38. **Euler Equations.** An equation of the form

$$t^2y'' + \alpha ty' + \beta y = 0, \qquad t > 0,$$

where α and β are real constants, is called an Euler equation. Show that the substitution $x = \ln t$ transforms an Euler equation into an equation with constant coefficients. Euler equations are discussed in detail in Section 5.5.

In each of Problems 39 through 42 use the result of Problem 38 to solve the given equation for $t > 0$.

39. $t^2y'' + ty' + y = 0$
40. $t^2y'' + 4ty' + 2y = 0$
41. $t^2y'' + 3ty' + 1.25y = 0$
42. $t^2y'' - 4ty' - 6y = 0$

3.5 Repeated Roots; Reduction of Order

In earlier sections we showed how to solve the equation

$$ay'' + by' + cy = 0 \qquad (1)$$

when the roots of the characteristic equation

$$ar^2 + br + c = 0 \qquad (2)$$

are either real and different, or are complex conjugates. Now we consider the third possibility, namely, that the two roots r_1 and r_2 are equal. This case occurs when the discriminant $b^2 - 4ac$ is zero, and it follows from the quadratic formula that

$$r_1 = r_2 = -b/2a. \qquad (3)$$

The difficulty is immediately apparent; both roots yield the same solution

$$y_1(t) = e^{-bt/2a} \qquad (4)$$

of the differential equation (1), and it is not obvious how to find a second solution.

EXAMPLE 1

Solve the differential equation

$$y'' + 4y' + 4y = 0. \qquad (5)$$

The characteristic equation is

$$r^2 + 4r + 4 = (r + 2)^2 = 0,$$

so $r_1 = r_2 = -2$. Therefore one solution of Eq. (5) is $y_1(t) = e^{-2t}$. To find the general solution of Eq. (5) we need a second solution that is not a multiple of y_1. This second solution can be found in several ways (see Problems 20 through 22); here we use a method originated by D'Alembert[5] in the eighteenth century. Recall that since $y_1(t)$ is a solution of Eq. (1), so is $cy_1(t)$ for any constant c. The basic idea is to generalize this observation by replacing c by a function $v(t)$ and then trying to determine $v(t)$ so that the product $v(t)y_1(t)$ is a solution of Eq. (1).

To carry out this program we substitute $y = v(t)y_1(t)$ in Eq. (1) and use the resulting equation to find $v(t)$. Starting with

$$y = v(t)y_1(t) = v(t)e^{-2t}, \qquad (6)$$

we have

$$y' = v'(t)e^{-2t} - 2v(t)e^{-2t} \qquad (7)$$

and

$$y'' = v''(t)e^{-2t} - 4v'(t)e^{-2t} + 4v(t)e^{-2t}. \qquad (8)$$

By substituting the expressions in Eqs. (6), (7), and (8) in Eq. (5) and collecting terms, we obtain

$$[v''(t) - 4v'(t) + 4v(t) + 4v'(t) - 8v(t) + 4v(t)]e^{-2t} = 0,$$

which simplifies to

$$v''(t) = 0. \qquad (9)$$

[5] Jean d'Alembert (1717–1783), a French mathematician, was a contemporary of Euler and Daniel Bernoulli, and is known primarily for his work in mechanics and differential equations. D'Alembert's principle in mechanics and d'Alembert's paradox in hydrodynamics are named for him, and the wave equation first appeared in his paper on vibrating strings in 1747. In his later years he devoted himself primarily to philosophy and to his duties as science editor of Diderot's Encyclopédie.

Therefore

$$v'(t) = c_1$$

and

$$v(t) = c_1 t + c_2, \tag{10}$$

where c_1 and c_2 are arbitrary constants. Finally, substituting for $v(t)$ in Eq. (6), we obtain

$$y = c_1 t e^{-2t} + c_2 e^{-2t}. \tag{11}$$

The second term on the right side of Eq. (11) corresponds to the original solution $y_1(t) = \exp(-2t)$, but the first term arises from a second solution, namely $y_2(t) = t\exp(-2t)$. These two solutions are obviously not proportional, but we can verify that they are linearly independent by calculating their Wronskian:

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix}$$
$$= e^{-4t} - 2te^{-4t} + 2te^{-4t} = e^{-4t} \neq 0.$$

Therefore

$$y_1(t) = e^{-2t}, \qquad y_2(t) = te^{-2t} \tag{12}$$

form a fundamental set of solutions of Eq. (5), and the general solution of that equation is given by Eq. (11). Note that both $y_1(t)$ and $y_2(t)$ tend to zero as $t \to \infty$; consequently, all solutions of Eq. (5) behave in this way. The graph of a typical solution is shown in Figure 3.5.1.

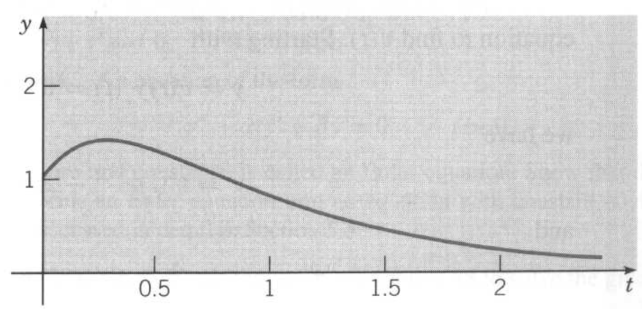

FIGURE 3.5.1 A typical solution of $y'' + 4y' + 4y = 0$.

The procedure used in Example 1 can be extended to a general equation whose characteristic equation has repeated roots. That is, we assume that the coefficients in Eq. (1) satisfy $b^2 - 4ac = 0$, in which case

$$y_1(t) = e^{-bt/2a}$$

is a solution. Then we assume that

$$y = v(t)y_1(t) = v(t)e^{-bt/2a} \tag{13}$$

and substitute in Eq. (1) to determine $v(t)$. We have

$$y' = v'(t)e^{-bt/2a} - \frac{b}{2a}v(t)e^{-bt/2a} \tag{14}$$

and

$$y'' = v''(t)e^{-bt/2a} - \frac{b}{a}v'(t)e^{-bt/2a} + \frac{b^2}{4a^2}v(t)e^{-bt/2a}. \tag{15}$$

Then, by substituting in Eq. (1), we obtain

$$\left\{a\left[v''(t) - \frac{b}{a}v'(t) + \frac{b^2}{4a^2}v(t)\right] + b\left[v'(t) - \frac{b}{2a}v(t)\right] + cv(t)\right\} e^{-bt/2a} = 0. \tag{16}$$

Canceling the factor $\exp(-bt/2a)$, which is nonzero, and rearranging the remaining terms, we find that

$$av''(t) + (-b+b)v'(t) + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)v(t) = 0. \tag{17}$$

The term involving $v'(t)$ is obviously zero. Further, the coefficient of $v(t)$ is $c - (b^2/4a)$, which is also zero because $b^2 - 4ac = 0$ in the problem that we are considering. Thus, just as in Example 1, Eq. (17) reduces to

$$v''(t) = 0;$$

therefore,

$$v(t) = c_1 t + c_2.$$

Hence, from Eq. (13), we have

$$y = c_1 te^{-bt/2a} + c_2 e^{-bt/2a}. \tag{18}$$

Thus y is a linear combination of the two solutions

$$y_1(t) = e^{-bt/2a}, \qquad y_2(t) = te^{-bt/2a}. \tag{19}$$

The Wronskian of these two solutions is

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-bt/2a} & te^{-bt/2a} \\ -\dfrac{b}{2a}e^{-bt/2a} & \left(1 - \dfrac{bt}{2a}\right)e^{-bt/2a} \end{vmatrix} = e^{-bt/a}. \tag{20}$$

Since $W(y_1, y_2)(t)$ is never zero, the solutions y_1 and y_2 given by Eq. (19) are a fundamental set of solutions. Further, Eq. (18) is the general solution of Eq. (1) when the roots of the characteristic equation are equal. In other words, in this case, there is one exponential solution corresponding to the repeated root, while a second solution is obtained by multiplying the exponential solution by t.

EXAMPLE 2

Find the solution of the initial value problem

$$y'' - y' + 0.25y = 0, \qquad y(0) = 2, \quad y'(0) = \tfrac{1}{3}. \tag{21}$$

The characteristic equation is

$$r^2 - r + 0.25 = 0,$$

so the roots are $r_1 = r_2 = 1/2$. Thus the general solution of the differential equation is

$$y = c_1 e^{t/2} + c_2 t e^{t/2}. \tag{22}$$

The first initial condition requires that

$$y(0) = c_1 = 2.$$

To satisfy the second initial condition, we first differentiate Eq. (22) and then set $t = 0$. This gives

$$y'(0) = \tfrac{1}{2}c_1 + c_2 = \tfrac{1}{3},$$

so $c_2 = -2/3$. Thus, the solution of the initial value problem is

$$y = 2e^{t/2} - \tfrac{2}{3}te^{t/2}. \tag{23}$$

The graph of this solution is shown in Figure 3.5.2.

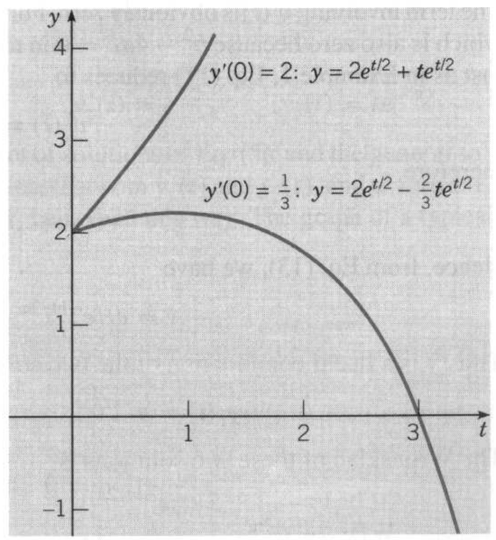

FIGURE 3.5.2 Solutions of $y'' - y' + 0.25y = 0$, $y(0) = 2$, with $y'(0) = 1/3$ and with $y'(0) = 2$, respectively.

Let us now modify the initial value problem (21) by changing the initial slope; to be specific, let the second initial condition be $y'(0) = 2$. The solution of this modified problem is

$$y = 2e^{t/2} + te^{t/2}$$

and its graph is also shown in Figure 3.5.2. The graphs shown in this figure suggest that there is a critical initial slope, with a value between $\frac{1}{3}$ and 2, that separates solutions that grow positively from those that ultimately grow negatively. In Problem 16 you are asked to determine this critical initial slope.

The geometrical behavior of solutions is similar in this case to that when the roots are real and different. If the exponents are either positive or negative, then the magnitude of the solution grows or decays accordingly; the linear factor t has little influence. A decaying solution is shown in Figure 3.5.1 and growing solutions in Figure 3.5.2. However, if the repeated root is zero, then the differential equation is $y'' = 0$ and the general solution is a linear function of t.

Summary. We can now summarize the results that we have obtained for second order linear homogeneous equations with constant coefficients,

$$ay'' + by' + cy = 0. \tag{1}$$

Let r_1 and r_2 be the roots of the corresponding characteristic polynomial

$$ar^2 + br + c = 0. \tag{2}$$

If r_1 and r_2 are real but not equal, then the general solution of the differential equation (1) is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \tag{24}$$

If r_1 and r_2 are complex conjugates $\lambda \pm i\mu$, then the general solution is

$$y = c_1 e^{\lambda t}\cos \mu t + c_2 e^{\lambda t}\sin \mu t. \tag{25}$$

If $r_1 = r_2$, then the general solution is

$$y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}. \tag{26}$$

Reduction of Order. It is worth noting that the procedure used earlier in this section for equations with constant coefficients is more generally applicable. Suppose we know one solution $y_1(t)$, not everywhere zero, of

$$y'' + p(t)y' + q(t)y = 0. \tag{27}$$

To find a second solution, let

$$y = v(t)y_1(t); \tag{28}$$

then

$$y' = v'(t)y_1(t) + v(t)y_1'(t)$$

and

$$y'' = v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t).$$

Substituting for y, y', and y'' in Eq. (27) and collecting terms, we find that

$$y_1 v'' + (2y_1' + py_1)v' + (y_1'' + py_1' + qy_1)v = 0. \tag{29}$$

Since y_1 is a solution of Eq. (27), the coefficient of v in Eq. (29) is zero, so that Eq. (29) becomes

$$y_1 v'' + (2y_1' + py_1)v' = 0. \tag{30}$$

Despite its appearance, Eq. (30) is actually a first order equation for the function v' and can be solved either as a first order linear equation or as a separable equation. Once v' has been found, then v is obtained by an integration. Finally, y is determined from

Eq. (28). This procedure is called the method of reduction of order because the crucial step is the solution of a first order differential equation for v' rather than the original second order equation for y. Although it is possible to write down a formula for $v(t)$, we will instead illustrate how this method works by an example.

EXAMPLE 3

Given that $y_1(t) = t^{-1}$ is a solution of

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0, \tag{31}$$

find a second linearly independent solution.

We set $y = v(t)t^{-1}$; then

$$y' = v't^{-1} - vt^{-2}, \qquad y'' = v''t^{-1} - 2v't^{-2} + 2vt^{-3}.$$

Substituting for y, y', and y'' in Eq. (31) and collecting terms, we obtain

$$\begin{aligned} &2t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) + 3t(v't^{-1} - vt^{-2}) - vt^{-1} \\ &= 2tv'' + (-4+3)v' + (4t^{-1} - 3t^{-1} - t^{-1})v \\ &= 2tv'' - v' = 0. \end{aligned} \tag{32}$$

Note that the coefficient of v is zero, as it should be; this provides a useful check on our algebra.

Separating the variables in Eq. (32) and solving for $v'(t)$, we find that

$$v'(t) = ct^{1/2};$$

then

$$v(t) = \tfrac{2}{3}ct^{3/2} + k.$$

It follows that

$$y = \tfrac{2}{3}ct^{1/2} + kt^{-1}, \tag{33}$$

where c and k are arbitrary constants. The second term on the right side of Eq. (33) is a multiple of $y_1(t)$ and can be dropped, but the first term provides a new independent solution. Neglecting the arbitrary multiplicative constant, we have $y_2(t) = t^{1/2}$.

PROBLEMS

In each of Problems 1 through 10 find the general solution of the given differential equation.

1. $y'' - 2y' + y = 0$
2. $9y'' + 6y' + y = 0$
3. $4y'' - 4y' - 3y = 0$
4. $4y'' + 12y' + 9y = 0$
5. $y'' - 2y' + 10y = 0$
6. $y'' - 6y' + 9y = 0$
7. $4y'' + 17y' + 4y = 0$
8. $16y'' + 24y' + 9y = 0$
9. $25y'' - 20y' + 4y = 0$
10. $2y'' + 2y' + y = 0$

In each of Problems 11 through 14 solve the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

11. $9y'' - 12y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = -1$
12. $y'' - 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 2$
13. $9y'' + 6y' + 82y = 0, \quad y(0) = -1, \quad y'(0) = 2$
14. $y'' + 4y' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 1$

▶ 15. Consider the initial value problem

$$4y'' + 12y' + 9y = 0, \qquad y(0) = 1, \quad y'(0) = -4.$$

(a) Solve the initial value problem and plot its solution for $0 \le t \le 5$.
(b) Determine where the solution has the value zero.
(c) Determine the coordinates (t_0, y_0) of the minimum point.
(d) Change the second initial condition to $y'(0) = b$ and find the solution as a function of b. Then find the critical value of b that separates solutions that always remain positive from those that eventually become negative.

16. Consider the following modification of the initial value problem in Example 2:

$$y'' - y' + 0.25y = 0, \qquad y(0) = 2, \quad y'(0) = b.$$

Find the solution as a function of b and then determine the critical value of b that separates solutions that grow positively from those that eventually grow negatively.

▶ 17. Consider the initial value problem

$$4y'' + 4y' + y = 0, \qquad y(0) = 1, \quad y'(0) = 2.$$

(a) Solve the initial value problem and plot the solution.
(b) Determine the coordinates (t_0, y_0) of the maximum point.
(c) Change the second initial condition to $y'(0) = b > 0$ and find the solution as a function of b.
(d) Find the coordinates (t_M, y_M) of the maximum point in terms of b. Describe the dependence of t_M and y_M on b as b increases.

18. Consider the initial value problem

$$9y'' + 12y' + 4y = 0, \qquad y(0) = a > 0, \quad y'(0) = -1.$$

(a) Solve the initial value problem.
(b) Find the critical value of a that separates solutions that become negative from those that are always positive.

19. If the roots of the characteristic equation are real, show that a solution of $ay'' + by' + cy = 0$ can take on the value zero at most once.

Problems 20 through 22 indicate other ways of finding the second solution when the characteristic equation has repeated roots.

20. (a) Consider the equation $y'' + 2ay' + a^2y = 0$. Show that the roots of the characteristic equation are $r_1 = r_2 = -a$, so that one solution of the equation is e^{-at}.
(b) Use Abel's formula [Eq. (8) of Section 3.3] to show that the Wronskian of any two solutions of the given equation is

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = c_1e^{-2at},$$

where c_1 is a constant.
(c) Let $y_1(t) = e^{-at}$ and use the result of part (b) to show that a second solution is $y_2(t) = te^{-at}$.

21. Suppose that r_1 and r_2 are roots of $ar^2 + br + c = 0$ and that $r_1 \ne r_2$; then $\exp(r_1t)$ and $\exp(r_2t)$ are solutions of the differential equation $ay'' + by' + cy = 0$. Show that $\phi(t; r_1, r_2) = [\exp(r_2t) - \exp(r_1t)]/(r_2 - r_1)$ is also a solution of the equation for $r_2 \ne r_1$. Then think of r_1 as fixed and use L'Hospital's rule to evaluate the limit of $\phi(t; r_1, r_2)$ as $r_2 \to r_1$, thereby obtaining the second solution in the case of equal roots.

22. (a) If $ar^2 + br + c = 0$ has equal roots r_1, show that

$$L[e^{rt}] = a(e^{rt})'' + b(e^{rt})' + ce^{rt} = a(r - r_1)^2e^{rt}. \tag{i}$$

Since the right side of Eq. (i) is zero when $r = r_1$, it follows that $\exp(r_1 t)$ is a solution of $L[y] = ay'' + by' + cy = 0$.

(b) Differentiate Eq. (i) with respect to r and interchange differentiation with respect to r and with respect to t, thus showing that

$$\frac{\partial}{\partial r}L[e^{rt}] = L\left[\frac{\partial}{\partial r}e^{rt}\right] = L[te^{rt}] = ate^{rt}(r - r_1)^2 + 2ae^{rt}(r - r_1). \quad \text{(ii)}$$

Since the right side of Eq. (ii) is zero when $r = r_1$, conclude that $t\exp(r_1 t)$ is also a solution of $L[y] = 0$.

In each of Problems 23 through 30 use the method of reduction of order to find a second solution of the given differential equation.

23. $t^2y'' - 4ty' + 6y = 0, \quad t > 0; \quad y_1(t) = t^2$
24. $t^2y'' + 2ty' - 2y = 0, \quad t > 0; \quad y_1(t) = t$
25. $t^2y'' + 3ty' + y = 0, \quad t > 0; \quad y_1(t) = t^{-1}$
26. $t^2y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0; \quad y_1(t) = t$
27. $xy'' - y' + 4x^3y = 0, \quad x > 0; \quad y_1(x) = \sin x^2$
28. $(x-1)y'' - xy' + y = 0, \quad x > 1; \quad y_1(x) = e^x$
29. $x^2y'' - (x - 0.1875)y = 0, \quad x > 0; \quad y_1(x) = x^{1/4}e^{2\sqrt{x}}$
30. $x^2y'' + xy' + (x^2 - 0.25)y = 0, \quad x > 0; \quad y_1(x) = x^{-1/2}\sin x$
31. The differential equation

$$xy'' - (x + N)y' + Ny = 0,$$

where N is a nonnegative integer, has been discussed by several authors.[6] One reason it is interesting is that it has an exponential solution and a polynomial solution.

(a) Verify that one solution is $y_1(x) = e^x$.

(b) Show that a second solution has the form $y_2(x) = ce^x \int x^N e^{-x}\,dx$. Calculate $y_2(x)$ for $N = 1$ and $N = 2$; convince yourself that, with $c = -1/N!$,

$$y_2(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^N}{N!}.$$

Note that $y_2(x)$ is exactly the first $N + 1$ terms in the Taylor series about $x = 0$ for e^x, that is, for $y_1(x)$.

32. The differential equation

$$y'' + \delta(xy' + y) = 0$$

arises in the study of the turbulent flow of a uniform stream past a circular cylinder. Verify that $y_1(x) = \exp(-\delta x^2/2)$ is one solution and then find the general solution in the form of an integral.

33. The method of Problem 20 can be extended to second order equations with variable coefficients. If y_1 is a known nonvanishing solution of $y'' + p(t)y' + q(t)y = 0$, show that a second solution y_2 satisfies $(y_2/y_1)' = W(y_1, y_2)/y_1^2$, where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2. Then use Abel's formula [Eq. (8) of Section 3.3] to determine y_2.

In each of Problems 34 through 37 use the method of Problem 33 to find a second independent solution of the given equation.

34. $t^2y'' + 3ty' + y = 0, \quad t > 0; \quad y_1(t) = t^{-1}$
35. $ty'' - y' + 4t^3y = 0, \quad t > 0; \quad y_1(t) = \sin(t^2)$

[6] T. A. Newton, "On Using a Differential Equation to Generate Polynomials," *American Mathematical Monthly* 81 (1974), pp. 592–601. Also see the references given there.

36. $(x-1)y'' - xy' + y = 0, \quad x > 1; \quad y_1(x) = e^x$
37. $x^2y'' + xy' + (x^2 - 0.25)y = 0, \quad x > 0; \quad y_1(x) = x^{-1/2}\sin x$

Behavior of Solutions as $t \to \infty$. Problems 38 through 40 are concerned with the behavior of solutions in the limit as $t \to \infty$.

38. If a, b, and c are positive constants, show that all solutions of $ay'' + by' + cy = 0$ approach zero as $t \to \infty$.
39. (a) If $a > 0$ and $c > 0$, but $b = 0$, show that the result of Problem 38 is no longer true, but that all solutions are bounded as $t \to \infty$.
(b) If $a > 0$ and $b > 0$, but $c = 0$, show that the result of Problem 38 is no longer true, but that all solutions approach a constant that depends on the initial conditions as $t \to \infty$. Determine this constant for the initial conditions $y(0) = y_0, y'(0) = y_0'$.
40. Show that $y = \sin t$ is a solution of

$$y'' + (k\sin^2 t)y' + (1 - k\cos t\sin t)y = 0$$

for any value of the constant k. If $0 < k < 2$, show that $1 - k\cos t\sin t > 0$ and $k\sin^2 t \geq 0$. Thus observe that even though the coefficients of this variable coefficient differential equation are nonnegative (and the coefficient of y' is zero only at the points $t = 0, \pi, 2\pi, \ldots$), it has a solution that does not approach zero as $t \to \infty$. Compare this situation with the result of Problem 38. Thus we observe a not unusual situation in the theory of differential equations: equations that are apparently very similar can have quite different properties.

Euler Equations. Use the substitution introduced in Problem 38 in Section 3.4 to solve each of the equations in Problems 41 and 42.

41. $t^2y'' - 3ty' + 4y = 0, \quad t > 0$
42. $t^2y'' + 2ty' + 0.25y = 0, \quad t > 0$

3.6 Nonhomogeneous Equations; Method of Undetermined Coefficients

We now return to the nonhomogeneous equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \quad (1)$$

where p, q, and g are given (continuous) functions on the open interval I. The equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \quad (2)$$

in which $g(t) = 0$ and p and q are the same as in Eq. (1), is called the homogeneous equation corresponding to Eq. (1). The following two results describe the structure of solutions of the nonhomogeneous equation (1) and provide a basis for constructing its general solution.