MAT 257Y Solutions to Practice Term Test 1

(1) Find the partial derivatives of the following functions

(a) $f(x, y, z) = \sin(x \sin(y \sin z))$

(b) $f(x, y, z) = x^{yz^2}$

Solution

(a) $\frac{\partial f}{\partial x}(x, y, z) = (\cos(x \sin(y \sin z)))(\sin(y \sin z))$
$\frac{\partial f}{\partial y}(x, y, z) = (\cos(x \sin(y \sin z)))(x \cos(y \sin z)) \sin z$
$\frac{\partial f}{\partial z}(x, y, z) = (\cos(x \sin(y \sin z)))(x \cos(y \sin z))y \cos z$

(b) First, we rewrite $f(x, y, z)$ as $f(x, y, z) = (e^{\ln x})^{yz^2} = e^{(\ln x)yz^2}$
$\frac{\partial f}{\partial x}(x, y, z) = (e^{(\ln x)yz^2})\frac{yz^2}{x} = (x^{yz^2})\frac{yz^2}{x}$
$\frac{\partial f}{\partial y}(x, y, z) = (e^{(\ln x)yz^2})(\ln x)z^2 = (x^{yz^2})(\ln x)z^2$
$\frac{\partial f}{\partial z}(x, y, z) = (e^{(\ln x)yz^2})(\ln x)y(2z) = (x^{yz^2})(\ln x)y(2z)$

(2) give an example of a nonempty set A such that the set of limit points of A is the same as the set of boundary points of A.

Solution

Let $A = S^1 = \{x \in R^2 | |x| = 1\}$. Then $A = Lim A = br(A)$.

(3) Let $A, B \subset R^n$ be compact.

Prove that the set $A + B = \{a + b | a \in A, b \in B\}$ is compact.

Solution

Consider the map $f\colon R^{2n} = R^n \times R^n \to R^n$ given by $f(x, y) = x + y$. This map is linear and hence continuous. By construction, $A + B = f(A \times B)$. $A \times B$ is compact as a product of two compact sets and hence $A + B = f(A \times B)$ is also compact as an image of a compact set under a continuous map.

(4) show that the intersection of arbitrary collection of closed sets is closed.

Solution

Let $\{A_\alpha\}_{\alpha\in I}$ be a collection of closed sets in R^n. Let $U_\alpha = R^n \backslash A_\alpha$. Then U_α is open.

We have

$$R^n \backslash \cap_\alpha A_\alpha = \cup_\alpha U_\alpha$$

is open as a union of open sets. Hence $\cap_\alpha A_\alpha$ is closed.

(5) show that $f\colon R^n \to R^m$ is continuous if and only if $f^{-1}(A)$ is closed for any closed $A \subset R^m$.

Solution

Let f be continuous.

Suppose $A \subset R^m$ is closed. Then $R^m \backslash A$ is open. By continuity of f this implies that $f^{-1}(R^m \backslash A)$ is open. It's easy to see that $f^{-1}(R^m \backslash A) = R^n \backslash f^{-1}(A)$. hence $f^{-1}(A)$ is closed. The reverse implication is proved similarly.

(6) Let R^{n^2} be the space of all $n \times n$ matrices. Consider the map $f\colon R^{n^2} \to R^{n^2}$ given by the formula

$f(A) = A \cdot A^T$.

Here A^T means the transpose of A.

Show that f is differentiable everywhere and compute $df(A)$.

Hint: use that $df(A)(X) = D_X f(A)$.

Solution

First observe that f is clearly differentiable because its components are polynomials in entries of A. to compute $df(A)$ we use the fact that for differentiable maps $df(A)(X) = D_X f(A)$.

By definition

$$D_X f(A) = \lim_{t\to 0} \frac{f(A+tX)-f(A)}{t} = \lim_{t\to 0} \frac{(A+tX)(A+tX)^T - AA^T}{t}$$

$$= \lim_{t\to 0} \frac{AA^T + tXA^T + tAX^T + t^2XX^T - AA^T}{t} = XA^T + AX^T$$

therefore $df(A)(X) = XA^T + AX^T$.

(7) Let $f = (f_1, f_2)\colon R^2 \to R^2$ be given by the formula $f_1(x,y) = x + y + y^3 + 1,\ f_2(x,y) = xe^y + 2$

Show that there exists an open set U containing $(0,0)$ such that $f\colon U \to f(U)$ is a bijection and f^{-1} is differentiable on $f(U)$ and compute $df^{-1}(1,2)$.

Solution

Clearly f is differentiable everywhere. we compute $\frac{\partial f_1}{\partial x}(x,y) = 1$, $\frac{\partial f_1}{\partial y}(x,y) = 1+3y^2$, $\frac{\partial f_2}{\partial x}(x,y) = e^y$, $\frac{\partial f_2}{\partial y}(x,y) = xe^y$

Therefore

$$[df(0,0)] = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

This matrix has det $= -1 \neq 0$. $f(0,0) = (1,2)$. hence, by the inverse function theorem, there exists an open set U containing $(0,0)$ such that $f\colon U \to f(U)$ is a bijection and f^{-1} is differentiable on $f(U)$ and $[df^{-1}(1,2)] = [df(0,0)]^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

(8) Let $f(x,y) = x^y$ be defined on $U = \{(x,y) | x > 0\}$.

Verify that

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y)$$

Solution

First we rewrite $f(x,y) = e^{(\ln x)y}$. we compute $\frac{\partial f}{\partial x}(x,y) = e^{(\ln x)y}\frac{y}{x}$, $\frac{\partial f}{\partial y}(x,y) = e^{(\ln x)y}\ln x$. Hence $\frac{\partial^2 f}{\partial x \partial y}(x,y) = e^{(\ln x)y}\frac{y}{x}\ln x + e^{(\ln x)y}\frac{1}{x}$ and $\frac{\partial^2 f}{\partial y \partial x}(x,y) = e^{(\ln x)y}\ln x\frac{y}{x} + e^{(\ln x)y}\frac{1}{x}$. Thus

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y)$$