MAT 257Y **Solutions to Term Test 1**

(1) (15 pts) Give the definitions of the following notions.

(a) an open set in R^n;

(b) a boundary point of a set $A \subset R^n$;

(c) a function $f\colon R^n \to R^m$ differentiable at a point p;

(d) a directional derivative of a function $f\colon R^n \to R^m$ at a point p.

Solution

(a) A set $U \subset R^n$ is called open if for every $p = (p_1, p_2, \dots, p_n) \in U$ there exists $\epsilon > 0$ such that the rectangle $I = (p_1 - \epsilon, p_1 + \epsilon) \times (p_1 - \epsilon, p_2 + \epsilon) \times \ldots \times (p_n - \epsilon, p_n + \epsilon)$ is contained in U.

(b) a point p is called a boundary point of A if for any $\epsilon > 0$ there exist $a \in B(p,\epsilon) \cap A$ and $b \in B(p,\epsilon) \cap A^c$

(c) a function $f\colon R^n \to R^m$ is differentiable at a point p if there exists a linear map $T\colon R^n \to R^m$ such that

$$\lim_{h\to 0} \frac{f(p+h) - f(p) - T(h)}{|h|} = 0$$

(d) Let $X \in R^n$ and let $g(t) = f(p + tX)$. Then $D_X f(p) = g'(0)$ if it exists is called the directional derivative of f at p in the direction X.

(2) (15 pts) Find the partial derivatives of the following functions

(a)

$$f(x,y) = \int_x^{\int_x^y g(t)dt} g(t)dt$$

Hint: put $F(x,y) = \int_x^y g(t)dt$ and express f as a composition.

(b) $f(x,y) = \ln((\sin(x+y^2))^{\cos 2x})$

Solution

(a) put $F(x,y) = \int_x^y g(t)dt$. By the fundamental theorem of calculus we have

$$\frac{\partial F}{\partial x}(x,y)(x,y) = -g(x) \text{ and } \frac{\partial F}{\partial y}(x,y)(x,y) = g(y)$$

We also have that $f(x,y) = F(x, F(x,y))$. Therefore, by the chain rule we have

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial F}{\partial x}(x,F(x,y))\frac{\partial x}{\partial x}(x,y) + \frac{\partial F}{\partial y}(x,F(x,y))\frac{\partial F}{\partial x}(x,y) =$$

$$= -g(x)\cdot 1 + g(F(x,y))\cdot(-g(x)) = -g(x) - g(\int_x^y g(t)dt)g(x)$$

Similarly,

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial F}{\partial x}(x,F(x,y))\frac{\partial x}{\partial y}(x,y) + \frac{\partial F}{\partial y}(x,F(x,y))\frac{\partial F}{\partial y}(x,y) =$$

$$= -g(x)\cdot 0 + g(F(x,y))g(y) = g(\int_x^y g(t)dt)g(y)$$

(b) First we simplify $f(x,y) = \ln((\sin(x+y^2))^{\cos 2x}) = (\cos 2x)\ln(\sin(x+y^2))$
Then we compute
$\frac{\partial f}{\partial x}(x,y) = -2(\sin 2x)\ln(\sin(x+y^2)) + (\cos 2x)\frac{1}{\sin(x+y^2)}\cos(x+y^2)$
$\frac{\partial f}{\partial y}(x,y) = (\cos 2x)\frac{1}{\sin(x+y^2)}\cos(x+y^2)(2y)$

(3) (20 pts) Let $f\colon R^2 \to R$ be given by the formula

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} \text{ if } (x,y) \neq (0,0) \\ 0 \text{ if } (x,y) = (0,0) \end{cases}$$

(a) Show that $f(x,y)$ is continuous at $(0,0)$.
(b) Show that f has partial derivatives at $(0,0)$.

(c) Does f has directional derivatives at $(0,0)$ in all directions?
(d) Show that f is not differentiable at $(0,0)$.

Solution

(a) we rewrite $f(x,y) = y\frac{x^2}{x^2+y^2}$. Clearly $|\frac{x^2}{x^2+y^2}| \leq 1$ and $\lim_{(x,y)\to(0,0)} y = 0$.
therefore $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.
(b) by definition of f we see that $f(x,0) = 0$ and $f(0,y) = 0$. therefore $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$.
(c) Does f has directional derivatives at $(0,0)$ in all directions?
For a direction $v = (v_1, v_2) \neq 0$ we compute $f((0,0)+tv) = f(tv_1, tv_2) = \frac{t^3 v_1^2 v_2}{t^2(v_1^2+v_2^2)} = t\frac{v_1^2 v_2}{(v_1^2+v_2^2)}$. This function is differentiable in t with $f((0,0)+tv)'(0) = \frac{v_1^2 v_2}{(v_1^2+v_2^2)}$. By definition this means that $D_v f(0,0)$ exists and is equal to $\frac{v_1^2 v_2}{(v_1^2+v_2^2)}$.
Answer: Yes.
(d) Suppose f is differentiable at $(0,0)$. then the matrix $[df(0,0)] = [\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)] = [0,0]$. By definition of differentiability this would mean that

$$\lim_{h\to 0} \frac{f(h) - f(0) - 0}{|h|} = 0$$

However along the line (t,t) we have

$$\lim_{t\to 0} \frac{f(t,t) - f(0,0)}{|t|} = \lim_{t\to 0} \frac{\frac{t^3}{2t^2}}{|t|} \neq 0$$

This is a contradiction which means that f is not differentiable at $(0,0)$.

(4) (10 pts) Show that a compact subset of R^n is bounded.

Solution

Let $C \subset R^n$ be compact. Let $U_n = B(0,n)$ where $n = 1,2,3\ldots$. Then U_n is open and $\cup_n U_n = R^n$.

Hence $C \subset \cup_n U_n$. By definition of compactness we can choose a finite subcover $U_{n_1}, \dots, U_{n_k}$ still covering C. Let $m = \max_k n_k$. Then $C \subset U_m$ and hence it is bounded.

(5) (10 pts) let $f(x, y) = x^2 + 5y^2 - 4xy - 2y$. Find all possible points of minimum of $f(x, y)$.

Solution

f is clearly differentiable everywhere. Its minimum can occur only at points where both partial derivatives vanish. we compute

$\frac{\partial f}{\partial x}(x, y) = 2x - 4y \quad \frac{\partial f}{\partial y}(x, y) = 10y - 4x - 2$

we solve

$$\begin{cases} 2x - 4y = 0 \\ 10y - 4x - 2 = 0 \end{cases} \qquad \begin{cases} x = 2 \\ y = 1 \end{cases}$$

Thus the only possible point of minimum is (2,1).

(6) (15 pts) Let $f\colon R^n \to R^m$ be continuous.

Are the following statements true or false? Prove if true and give counterexamples if false.

(a) If $A \subset R^n$ is closed and bounded then $f(A)$ is closed and bounded.
(b) If $A \subset R^n$ is closed then $f(A)$ is closed.
(c) If $A \subset R^n$ is bounded then $f(A)$ is bounded.

Solution

(a) **True**
If $A \subset R^n$ is closed and bounded if and only if it's compact and and image of a compact set under a continuous map is compact.
(b) **False**. let $f(x) = \arctan x$. Then $f([0, \infty)) = [0, \pi/2)$ is not closed.
(c) **True**
If A is bounded it's contained in a closed ball $B = \bar{B}(0, R) = \{x \in R^n \text{ such that } |x| \le R\}$ for some $R > 0$. Then $f(A) \subset f(B)$ but B is compact.

hence $f(B)$ is also compact and in particular it is bounded.

(7) (15 pts) Let $GL(n, R)$ be the set of all $n \times n$ invertible matrices.

Show that $GL(n, R)$ is open in R^{n^2}.

Solution

Lel $f\colon\ R^{n^2} \to R$ be given by $f(A) = \det(A)$. then f is a polynomial in coordinate entries and hence is continuous. We know that A is invertible if an only if $\det(A) \neq 0$.

Therefore $GL(n, R) = f^{-1}((-\infty, 0) \cup (0, \infty))$ and hence is open as the preimage of an open set.