Solutions to Term Test 3

(1) (12 pts) Give the following definitions

(a) k-volume of a parallelepiped $P(v_1, \ldots, v_k)$ for $v_1, \ldots, v_k \in \mathbb{R}^n$.

(b) A C^r-manifold without a boundary in $\mathbb{R}^n$.

Solution

(a) Let A be the matrix with columns $v_1, \ldots, v_k$. Then $P(v_1, \ldots, v_k) = \sqrt{\det(A^t A)}$

(b) A set $M \subset \mathbb{R}^n$ is a k-dimensional C^r-manifold without a boundary if for every point $p \in M$ there exists a set $U \subset M$ which is open in M, an open subset $V \subset \mathbb{R}^k$ and a C^r map $f\colon V \to \mathbb{R}^n$ such that

(i) $f(V) = U$ and $f\colon V \to U$ is 1-1 and onto;

(ii) $rank[df_x] = k$ for any $x \in V$;

(iii) $f^{-1}\colon U \to V$ is continuous.

(2) (15 pts) Let $w \in R^3$ be a fixed vector. Define $T_w(u, v) = \langle u \times v, w\rangle$.

(a) Prove that T_w is a 2-tensor on $\mathbb{R}^3$

(b) Let $w = (2, 1, -1)$. Express T_w in the standard basis of $\mathcal{L}^2(\mathbb{R}^3)$.

Solution

(a) This immediately follows from multi-linearity of cross product and scalar product.

(b) We have

$$T_w(u, v) = \det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ 2 & 1 & -1 \end{pmatrix} = 2(u_2v_3 - u_3v_2) - (u_1v_3 - u_3v_1) - (u_1v_2 - u_2v_1) =$$

$$= 2e_2^* \otimes e_3^* - 2e_3^* \otimes e_2^* - e_1^* \otimes e_3^* + e_3^* \otimes e_1^* - e_1^* \otimes e_2^* + e_2^* \otimes e_1^*$$

(3) (15 pts) Let $M \subset \mathbb{R}^3$ be given by $\{x^2 + y^2 - z^2 = 0\} \cap \{x + 2y - z = 1\}$.

Show that M is a manifold and compute its dimension.

Solution

Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(x,y,z) = (x^2 + y^2 - z^2, x + 2y - z)$. Note that f is clearly C^∞ and $M = f^{-1}\{(0,1)\}$.

We claim that $(0,1)$ is a regular value of f. To see this suppose $f(x,y,z) = (0,1)$. Then $[df_{(x,y,z)}] = \begin{pmatrix} 2x & 2y & -2z \\ 1 & 2 & -1 \end{pmatrix}$

The only way this matrix could have rank 1 if the first row is a multiple of the second one i.e. $x = \lambda, y = 2\lambda, z = \lambda$. Then we must have $0 = x^2 + y^2 - z^2 = \lambda^2 + (2\lambda)^2 - (\lambda)^2 = 4(\lambda)^2$ and hence $\lambda = 0$ and $x = y = z = 0$. This contradicts $x + 2y - z = 1$. Thus $[df_{(x,y,z)}]$ has rank=2. Therefore $(0,1)$ is a regular value of f and hence, M is a C^∞ manifold without boundary of dimension $3 - 2 = 1$.

(4) (15 pts) Let $U \subset \mathbb{R}^2$ be given by $\{0 < x^2 + 4y^2 < 1\}$ and $f(x,y) = \frac{1}{\sqrt{x^2+4y^2}}$.

Determine if $\int_U^{ext} f$ exists and if it does compute it.

Solution

Let $U_n = \{\frac{1}{n^2} < x^2 + 4y^2 < 1\}$ where $n > 1$ be an open exhaustion of U. Then U_n is rectifiable and f is continuous and bounded on U_n. Therefore, $\int_{U_n} f$ exists for any $n > 1$ and hence $\int_{U_n}^{ext} f$ exists and $\int_{U_n}^{ext} f = \int_{U_n} f$. Let $V_n = U_n \setminus \{(x,0) | x > 0\}$. Then V_n is also open and rectifiable and hence $\int_{V_n} f$ also exists. Since $f \cdot \chi_{U_n} = f \cdot \chi_{V_n}$ except on a set of measure zero this means that $\int_{V_n} f = \int_{U_n} f$.

To compute $\int_{V_n} f$ we use the change of variables $(x,y) = g(r,\theta)$ given by $x = r\cos\theta, y = \frac{1}{2} r \sin\theta$ where $1/n < r < 1, 0 < \theta < 2\pi$. Then $\det[dg] = \frac{r}{2}$

and $\int_{V_n} f = \int_0^{2\pi} \int_{1/n}^1 \frac{r}{2r} = \pi(1-1/n)$. We see that $\lim_{n\to\infty} \int_{U_n}^{ext} f = \lim_{n\to\infty} \pi(1-1/n) = \pi$. Since $f > 0$ on U this means that $\int_U^{ext} f$ exists and $\int_U^{ext} f = \pi$.

(5) (15 pts) Find the area of the following parametrized surface:

$\alpha(s,t) = (s\cos 2t, s\sin 2t, s)$ where $0 < s < 1, 0 < t < 2$.

Solution

Let $U = \{(s,t)|$ such that $0 < s < 1, 0 < t < 2$. By definition, $area(\alpha) = \int_U^{ext} \text{vol}_2 P(\frac{\partial\alpha}{\partial s}, \frac{\partial\alpha}{\partial t})$. We compute, $\frac{\partial\alpha}{\partial s} = (\cos 2t, \sin 2t, 1)^t, \frac{\partial\alpha}{\partial t} = (-2s\sin 2t, 2s\cos 2t, 0)^t$ and $\langle\frac{\partial\alpha}{\partial s}\frac{\partial\alpha}{\partial s}\rangle = 2, \langle\frac{\partial\alpha}{\partial s}\frac{\partial\alpha}{\partial t}\rangle = 0, \langle\frac{\partial\alpha}{\partial t}\frac{\partial\alpha}{\partial t}\rangle = 4s^2$. This means that $\text{vol}_2 P(\frac{\partial\alpha}{\partial s}, \frac{\partial\alpha}{\partial t}) = \sqrt{\det\begin{pmatrix} 2 & 0 \\ 0 & 4s^2 \end{pmatrix}} = 2\sqrt{2}s$.

Thus

$$area(\alpha) = \int_U^{ext} \text{vol}_2 P(\frac{\partial\alpha}{\partial s}, \frac{\partial\alpha}{\partial t}) = \int_U^{ext} 2\sqrt{2}s =$$

$$= \int_0^2 (\int_0^1 2\sqrt{2}s ds)dt = 2\sqrt{2}$$

(6) (10 pts) Let $v_1 = (1,1,0), v_2 = (-1,0,1), v_3 = (1,1,1)$ and $w_1 = (0,2,0), w_2 = (1,1,0), w_3 = (-2,1,3)$ be two bases of $\mathbb{R}^3$. Do (v_1, v_2, v_3) and (w_1, w_2, w_3) have the same orientation?

Solution

Consider the standard basis $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$. The transition matrix from (e_1, e_2, e_3) to (v_1, v_2, v_3) has columns v_1, v_2, v_3, i.e. it's given by $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. We compute $\det A = 1$. Hence, (e_1, e_2, e_3) and (v_1, v_2, v_3) have the same orientation.

Similarly, we compute that the transition matrix from (e_1, e_2, e_3) to (u_1, u_2, u_3) is $B = \begin{pmatrix} 0 & 1 & -2 \\ 2 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ and $\det B = -6$. Hence, (e_1, e_2, e_3) and (u_1, u_2, u_3) have opposite orientations. Therefore, (v_1, v_2, v_3) and (w_1, w_2, w_3) have opposite orientations.

(7) (18 pts)

Let $v_1, \dots, v_k \in \mathbb{R}^n$ where $n \geq k$.

(a) Let A be the matrix with columns $v_1, \dots, v_k$ and let Q be a $k \times k$ orthogonal matrix. Let $u_1, \dots, u_k$ be columns of AQ.
Prove that $\text{vol}_k P(v_1, \dots, v_k) = \text{vol}_k P(u_1, \dots, u_k)$

(b) Let $k \geq 2$.
Prove that $\text{vol}_k P(v_1, \dots, v_k) = \text{vol}_k P(-v_2, v_1, \dots, v_k)$.

Solution

(a) By definition, $\text{vol}_k P(v_1, \dots, v_k) = \sqrt{\det A^t A}$ and $\text{vol}_k P(u_1, \dots, u_k) = \sqrt{\det (AQ)^t (AQ)} = \sqrt{\det Q^t (A^t A) Q} = \sqrt{\det Q^t \cdot \det(A^t A) \cdot \det Q} = \sqrt{(\det Q)^2 \cdot \det(A^t A)} = \sqrt{\det A^t A} = \text{vol}_k P(v_1, \dots, v_k)$ where we used the fact that since Q is orthogonal, $\det Q = \pm 1$.

(b) Let Q be the $k \times k$ matrix with $Q_{2,1} = -1, Q_{1,2} = 1, Q_{i,i} = 1$ for $i \geq 3$ and $Q_{i,j} = 0$ for other values of (i, j). Then Q is clearly orthogonal and $(v_1 | v_2 | v_3 \dots | v_k) Q = (-v_2 | v_1 | v_3 | \dots | v_k)$ and the result follows by part a).