

- (1) (15 pts) Let (X, d) be a metric space. Let $A \subset X$ be a compact subset. Using only the definition of compactness prove that A is closed.

Solution

Let $U = X \setminus A$. We need to show that U is open. Let $p \in U$. Let $U_n = \{x \in X \mid \text{such that } d(x, p) > 1/n\}$. Then U_n is open. Indeed, let $x \in U_n$. Then $d(x, p) > 1/n$. Choose $\epsilon > 0$ such that $d(x, p) - \epsilon > 1/n$. Then $B(x, \epsilon) \subset U_n$ by the triangle inequality. Hence U_n is open. Next note that $\cup_{n \in \mathbb{N}} U_n = X \setminus \{p\}$ covers A . By compactness of A we can find a finite subcover U_{n_1}, \dots, U_{n_k} covering A where $n_1 < n_2 < \dots < n_k$. Then $\cup_{j=1}^k U_{n_j} = U_{n_k}$ contains A . This means that $B(p, \frac{1}{n_k}) \subset U$ and hence U is open.

- (2) (10 pts) Let $f: X \rightarrow \mathbb{R}$ be continuous at $a \in X$. Prove that there exists $\delta > 0$ such that f is bounded on $B(a, \delta)$.

Solution

Choose any $\epsilon > 0$. For example, take $\epsilon = 1$. By definition of continuity, there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ for any $x \in B(a, \delta)$. This means that for any $x \in B(a, \delta)$ we have $|f(x)| < |f(a)| + \epsilon$.
□

- (3) (15pts) Mark True or False. **If True give a proof, if False give a counterexample.**

Let (X, d) be a metric space. Let $A, B \subset X$ be subsets in X .

- (a) $\text{ext}(A)$ is open;
 (b) $\text{int}(A \cup B) = \text{int}(A) \cup \text{int}(B)$.

Solution

- (a) **True.** Let $p \in \text{ext}(A)$. By definition of $\text{ext}(A)$ this means that there exists $\epsilon > 0$ such that

$B(p, \epsilon) \subset X \setminus A$. We claim that $B(p, \epsilon) \subset \text{ext}(A)$. Let $x \in B(p, \epsilon)$. Then $d(p, x) < \epsilon$. Choose $\delta > 0$ such that $d(p, x) + \delta < \epsilon$. Then, by the triangle inequality, $B(x, \delta) \subset B(p, \epsilon) \subset X \setminus A$. This means that $B(p, \epsilon) \subset \text{ext}(A)$ and hence $\text{ext}(A)$ is open.

(b) **False.** For example, take $X = \mathbb{R}$, $A = [0, 1]$ and $B = [1, 2]$. then $\text{int}(A) = (0, 1)$, $\text{int}(B) = (1, 2)$ and $\text{int}(A \cup B) = (0, 2) \neq (0, 1) \cup (1, 2)$.

(4) (15 pts) Find expressions for the partial derivatives of the following functions

(a) $F(x, y) = f(g(x)k(y), h(x) + 2k(y))$

(b) $F(x, y) = \int_{k^2(x)h(y)}^1 g(t) dt$

Solution

(a) Let $f = f(t_1, t_2)$. Then, by the chain rule we have

$$\begin{aligned} \frac{\partial F}{\partial x}(x, y) &= \frac{\partial f}{\partial t_1}(g(x)k(y), h(x) + 2k(y)) \cdot \frac{\partial(g(x)k(y))}{\partial x} + \\ &+ \frac{\partial f}{\partial t_2}(g(x)k(y), h(x) + 2k(y)) \cdot \frac{\partial(h(x) + k(y))}{\partial x} = \\ &= \frac{\partial f}{\partial t_1}(g(x)k(y), h(x) + 2k(y)) \cdot g'(x)k(y) + \frac{\partial f}{\partial t_2}(g(x)k(y), h(x) + 2k(y)) \cdot h'(x) \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\partial F}{\partial x}(x, y) &= \\ &= \frac{\partial f}{\partial t_1}(g(x)k(y), h(x) + 2k(y)) \cdot g(x)k'(y) + \frac{\partial f}{\partial t_2}(g(x)k(y), h(x) + 2k(y)) \cdot 2k'(y) \end{aligned}$$

(b) Let $a(x, y) = k^2(x)h(y)$ and $f(a) = \int_a^1 g(t) dt$. Then $F(x, y) = f(a(x, y))$. Note that $f'(a) = -g(a)$ by the Fundamental Theorem of Calculus.

By the chain rule,

$$\begin{aligned}\frac{\partial F}{\partial x}(x, y) &= \frac{\partial f}{\partial a}(a(x, y)) \cdot \frac{\partial a}{\partial x} = f'(k^2(x)h(y)) \cdot 2k(x)h(y) = \\ &= -g(k^2(x)h(y)) \cdot 2k(x)h(y)\end{aligned}$$

Similarly,

$$\frac{\partial F}{\partial y}(x, y) = -g(k^2(x)h(y)) \cdot k^2(x)h'(y)$$

(5) (20 pts) Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

- (a) Is f continuous at $(0, 0)$?
- (b) Do D_1f and D_2f exist at $(0, 0)$?
- (c) Is f differentiable at $(0, 0)$?

Solution

- (a) Is f continuous at $(0, 0)$? We can rewrite $f(x, y) = x \cdot \frac{y}{\sqrt{x^2+y^2}}$ where $\lim_{(x,y) \rightarrow 0} x = 0$ and $|\frac{y}{\sqrt{x^2+y^2}}| \leq 1$. Therefore,

$$\lim_{(x,y) \rightarrow 0} f(x, y) = 0 = f(0, 0)$$

Which means that f is continuous at $(0, 0)$.

Answer: Yes.

- (b) Do D_1f and D_2f exist at $(0, 0)$?
Observe that $f(x, 0) = 0$ and $f(0, y) = 0$ for any x and any y . Therefore $D_1f(0, 0) = 0 = D_2f(0, 0)$.

Answer: Yes, both partial derivatives exist at $(0, 0)$.

- (c) Is f differentiable at $(0, 0)$? If f were differentiable at 0 then all directional derivatives at $(0, 0)$ would exist too. Let $v = (1, 1)$ Then $D_v f(0, 0) =$

$f(tv)'(0)$ but $f(tv) = \frac{t^2}{\sqrt{t^2+t^2}} = \frac{|t|^2}{\sqrt{2}|t|} = \frac{1}{\sqrt{2}}|t|$ has no derivative at 0. Hence $D_v f(0,0)$ does not exist and hence f is not differentiable at $(0,0)$.

Answer: No.

- (6) (10 pts) Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $f(x,y) = (xy, e^x + y)$.

Show that there exists an open set U containing $(0,1)$ such that $V = f(U)$ is open, f is 1-1 on U and $g = f^{-1}: V \rightarrow U$ is differentiable on V .

Compute $dg_{(0,2)}$.

Solution

F is clearly C^1 on \mathbb{R}^2 . We compute the matrix of $df_{(x,y)}$. It is given by

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} y & x \\ e^x & 1 \end{pmatrix}$$

Therefore, the matrix of $df_{(0,1)}$ is

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

It has $\det = 1 \neq 0$ and hence it's invertible. By the Inverse Function Theorem, there exists an open set U containing $(0,1)$ such that $V = f(U)$ is open, f is 1-1 on U and $g = f^{-1}: V \rightarrow U$ is differentiable on V . Observe that $f(0,1) = (0,2)$. Again, by the Inverse Function theorem, the matrix of $dg_{(0,2)}$ is given by

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

- (7) (15 pts) Let $M(n)$ be the set of all real $n \times n$ matrices identified with \mathbb{R}^{n^2} . Let $O(n) \subset M(n)$ be the set of all orthogonal matrices. Recall that an $n \times n$ matrix

is called orthogonal if $A \cdot A^t = A^t \cdot A = \text{Id}$ where A^t is the transpose of A .

- (a) Prove that $O(n)$ is closed.
- (b) Prove that $O(n)$ is bounded.

Solution

- (a) Consider the map $f: M(n) \rightarrow M(n)$ given by $f(A) = A \cdot A^t$. Then f is clearly continuous as the entries of $f(A)$ are polynomials in A_{ij} 's. By definition, $O(n) = f^{-1}(\{\text{Id}\})$. Since $\{\text{Id}\} \subset M(n)$ is closed we conclude that $O(n)$ is also closed as a preimage of a closed set under a continuous map.
- (b) Note that $\|A\| = \sqrt{\sum_{i,j} A_{i,j}^2}$ for any $A \in M(n)$. Let $A \in O(n)$. We are given that $A \cdot A^t = \text{Id}$. Observe that the i -th column of A^t is the i th row of A . Multiplying the i -th row of A by the i -th column of A^t this gives $1 = \sum_j A_{i,j}^2$. Adding the above equations for all i 's this gives $\sum_i \sum_j A_{i,j}^2 = n$. This means that $\|A\| = \sqrt{n}$ for any $A \in O(n)$ and hence $O(n)$ is bounded.
Note that taken together (a) and (b) mean that $O(n)$ is compact.