(1) Let Q be a rectangle in $\mathbb{R}^n$ and let $A \subset Q$ be a subrectangle. Let $f\colon Q \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

Prove that $\int_Q f$ exists and $\int_Q f = \text{vol(A)}$.

(2) A set $S \subset \mathbb{R}^n$ is said to have *content* zero if for any $\epsilon > 0$ there exists a finite collection of rectangles Q_i covering S such that $\sum_i \text{vol}(\text{Q}_\text{i}) < \epsilon$.

(a) Show that if $S \subset Q \subset \mathbb{R}^n$ has content zero then any bounded function $f\colon Q \to \mathbb{R}$ such that $f(x) = 0$ if $x \notin S$ is integrable over Q and $\int_Q f = 0$.

(b) Let $S \subset Q \subset \mathbb{R}^n$ have content zero. let $f, g\colon Q \to \mathbb{R}$ be bounded and satisfy $f(x) = g(x)$ if $x \notin S$. Prove that $\int_Q f$ exists if and only if $\int_Q g$ exists and if they both exist $\int_Q f = \int_Q g$.

(c) Show that a finite union of sets of content zero has content zero.

(d) let Q be a rectangle in $\mathbb{R}^n$ show that $bd(Q)$ has content zero.

(e) Show that if S has content zero then its closure $Cl(S)$ also has content zero.

(f) Show that $S = \mathbb{Q} \cap [0, 1] \subset \mathbb{R}$ does not have content zero. Here $\mathbb{Q}$ is the set of rational numbers.

(3) let $f\colon Q \to R$ be integrable over Q. let $c \in R$ be a constant. Prove that cf is also integrable over Q and $\int_Q cf = c \int_Q f$.

Note: The proof depends on the sign of c.

Consider the Cantor set S on $[0, 1]$ constructed as follows. Let S_1 be obtained from $[0, 1]$ by removing the open interval $(1/3, 2/3)$. Let S_2 be obtained from S_1 by further removing middle intervals $(1/9, 2/9)$ and $(7/9, 8/9)$ out of S_1 etc. Let $S = \cap_{i=1}^{\infty} S_i$ be the Cantor set. Show that S has content zero.

(4) Let $f\colon [0, 1] \to R$ be defined as follows

$$f(x) = \begin{cases} 1/q \text{ if } x = p/q \text{ where p,q are positive integers with no common factor} \\ 0 \text{ if } x \text{ is irrational} \end{cases}$$

Show that f is integrable on $[0, 1]$.

(5) Let $f\colon R^n \to R$ be integrable. The graph of f is the set $\Gamma_f = \{(x, y) \in R^{n+1} |$ such that $y = f(x)\}$.

Show that Γ_f has measure zero.

Hint: Use the definition of integrability of f.

(6) Let Q be a rectangle in R^n covered by countably many rectangles Q_i.

$$Q \subset \cup_{i=1}^{\infty} Q_i$$

Prove that

$$\mathrm{volQ} \leq \sum_{\mathrm{i=1}}^{\infty} \mathrm{volQ_i}$$

Hint: Substitute Q_i by slightly bigger rectangles Q_i' such that $Q_i \subset int(Q_i')$ and use compactness of Q.