(1) Give an ϵ-δ proof of the following statement:
Let $f\colon R^n \to R^m$ and $g\colon R^n \to R$ satisfy

$$\lim_{x \to a} f(x) = 0 \text{ and } g(x) \text{ is bounded on } B(a,\delta) \text{ for some } \delta > 0$$

then $\lim_{x \to a} g(x)f(x) = 0$

(2) Let $A = \mathbb{Q} \cap [0,1] \subset \mathbb{R}$. Give an example of an open cover of A which has no finite subcover.

(3) Give an example of a metric space X and a subset $A \subset X$ such that A is closed and bounded but not compact.

(4) Let (X,d) be a metric space. Let $A \subset X$ be a subset. Prove that $A \subset X$ is compact if and only if $A \subset (A, d_A)$ is compact. Recall that d_A is the restriction of the metric d to A.

(5) Let X be a metric space. Prove that the intersection of an arbitrary family of compact subsets of X is compact.

(6) Let $A = \{0, 1, 1/2, 1/3, \ldots, 1/n, \ldots\} \subset \mathbb{R}$.
Using only the definition of compactness prove that the A is compact.

(7) Let $(X, d_X), (Y, d_Y)$ be metric spaces. define $d_{prod}\colon (X \times Y) \times (X \times Y) \to \mathbb{R}$ by $d_{prod}((x_1,y_1),(x_2,y_2)) = \sqrt{d_X(x_1,x_2)^2 + d_Y(y_1,y_2)^2}$.
   (a) Prove that d_{prod} is a metric on $X \times Y$.
   (b) verify that if (X, d_X) is $\mathbb{R}^n$ with the standard metric and (Y, d_Y) is $\mathbb{R}^m$ with the standard metric then $(X \times Y, d_{prod})$ is $\mathbb{R}^{n+m}$ with the standard metric.

(8) Let (X,d) be a metric space. Let $A \subset X$ be any subset. let $f\colon X \to \mathbb{R}$ be defined by $f(x) = \inf_{y \in A} d(x,y)$. We'll refer to $f(x)$ as the distance from x to A and denote it by $d(x,A)$.
   (a) Prove that f satisfies $|f(x) - f(y)| \leq d(x,y)$ for any $x, y \in X$.
   (b) Prove that f is continuous on X.
   (c) Prove that $A \subset X$ is closed if an only if the following holds

$$d(x,A) = 0 \text{ if and only if } x \in A$$

   (d) Let $A \subset \mathbb{R}^2$ be the graph of $y = x^2$ where $x \in \mathbb{R}$.
   Let $p \in \mathbb{R}^2$ be the point $(0,1)$.
   Find $d(p,A)$. Here d is the standard metric on $\mathbb{R}^2$.

(9) Which of the following sets are compact? provide an explanation.
   (a) $\emptyset \subset \mathbb{R}$;
   (b) $\mathbb{Z} \subset \mathbb{R}$;
   (c) $\{(x,y) \in \mathbb{R}^2|$ such that $0 < x^2 + y^2 \leq 1\}$.
   (d) $\{(x,y) \in \mathbb{R}^2|$ such that $x \geq 1, 0 \leq y \leq \frac{1}{x}\}$.
   (e) $\{(x,y,z) \in \mathbb{R}^3|$ such that $x^2 + y^2 \leq z \leq 2x + 4y\}$.

Extra Credit Problem (to be written up and submitted separately)

A norm on a real vector space V is a function $|\cdot|\colon V \to \mathbb{R}$ satisfying the following conditions

(1) $|v| \geq 0$ for any $v \in V$ and $|v| = 0$ if and only if $v = 0$.
(2) $|\lambda v| = |\lambda| \cdot |v|$ for any $v \in V, \lambda \in \mathbb{R}$.
(3) $|v_1 + v_2| \leq |v_1| + |v_2|$ for any $v_1, v_2 \in V$. given a norm one can define a metric on V by the formula $d(u, v) = |u - v|$.

Prove that any two norms on $\mathbb{R}^n$ define the same open sets.

Hint: Use that the unit sphere in $\mathbb{R}^n$ is compact and hence any continuous function on $\mathbb{R}^n$ attains a maximum and a minimum on the unit sphere.