(1) Let $V \subset \mathbb{R}^n$ be a vector subspace of dimension $n-1$. Let $N \in \mathbb{R}^n$ be a nonzero vector normal to V. Let $v_1, \dots, v_{n-1}$ be a basis of V. We'll say that $(v_1, \dots, v_{n-1})$ is a positive basis with respect to the orientation of V induced by N if $\det(N, v_1, \dots, v_{n-1}) > 0$.

Prove that this defines a well-defined orientation of V. in other words, suppose $(u_1, \dots, u_{n-1})$ be another basis of V.

Prove that $(u_1, \dots, u_{n-1})$ and $(v_1, \dots, v_{n-1})$ have the same orientation if and only if $\det(N, v_1, \dots, v_{n-1})$ and $\det(N, u_1, \dots, u_{n-1})$ have the same sign.

(2) Let $M \subset \mathbb{R}^n$ be orientable and let μ be the orientation induced by a positive atlas f_α.

In other words for any point $p \in M$ and a positive parametrization $f_\alpha \colon V_\alpha \to U_\alpha$ such that $p = f_\alpha(q_\alpha)$ we choose the orientation of T_pM by declaring the basis of T_pM given by $df_{q_\alpha}(e_1), \dots, df_{q_\alpha}(e_k)$ to be positive.

Prove that μ is continuous. In other words, suppose $U \subset M$ is an open set and $X_1(x), \dots, X_k(x)$ be continuous vector fileds on U such that $(X_1(x), \dots, X_k(x))$ is a basis of T_xM. Define $h(x) = +1$ if $(X_1(x), \dots, X_k(x))$ is a positive basis of $T_x(M)$ with respect to μ and $h(x) = -1$ if $(X_1(x), \dots, X_k(x))$ is a negative basis of $T_x(M)$ with respect to μ.

Hint: Let $f_\alpha \colon V_\alpha \to U_\alpha$ be a positive parametrization. look at the vector fields $Y_1, \dots, Y_k$ on U_α given by $Y_i(f_\alpha(x)) = df_\alpha(x)(e_i)$. Then $Y_1(p), \dots, Y_k(p)$ is a positive basis of T_pM for any $p \in U_\alpha$ by definition of the orientation μ. Then for any $p \in U \cap U_\alpha$ we have that $(X_1(p), \dots, X_k(p))$ and $(Y_1(p), \dots, Y_k(p))$ are related by a $k \times k$ matrix $A(p)$ depending continuously on p. Use that $\det A(p)$ is continuous in p.

(3) Let $M = \{x^2 + y^2 + z^2 \leq 1\}$ in $\mathbb{R}^3$ with the orientation coming from the canonical orientation on $\mathbb{R}^3$. Consider the induced orientation on ∂M and find a positive basis of $T_p\partial M$ at $p = (1, 0, 0)$.

Further, let $N = S^2_+ = \{(x, y, z)|$ such that $x^2 + y^2 + z^2 = 1$ and $z \geq 0\}$. Consider the orientation on N coinciding with the orientation on $S^2 = \partial M$. Consider ∂N with the induced orientation from N. Find a positive basis of $T_p\partial N$ for $p = (1, 0, 0)$.

(4) Let $M_1 \subset \mathbb{R}^{n_1}, M_2 \subset \mathbb{R}^{n-2}$ be orientable manifolds without boundary. Prove that $M_1 \times M_2 \subset \mathbb{R}^{n_1+n_2}$ is orientable.

(5) Let $M^k \subset \mathbb{R}^n$ be a C^∞ manifold with boundary. Prove that for any $p \in \partial M$ there exists an open set $W \subset \partial M$ we can construct a C^∞ unit vector filed N on U tangent to ∂M such that $N(x) \perp T_x\partial M$ for any $x \in W$.

Hint: Take a local parametrization $f \colon V \to U$ where $V \subset H^k$, $U \subset M$ and look at the vector fields $df_x(e_1), \dots, df_x(e_k)$. Apply Gramm-Shmidt to those vector fields.

(6) Let $M = \{x^2/9 + y^2/4 + z^2 \leq 3\}$ in R^3. Consider the induced orientation on ∂M and find a positive basis of $T_p \partial M$ at $p = (3, -2, 1)$.

(7) Let $M = S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ with the orientation induced from the ball $\{x^2 + y^2 + z^2 \leq 1\}$. Let $\omega = z dx \wedge dy$. Compute $\int_M \omega$.

(8) Let M be the cylinder $\{(x, y, z) |$ such that $x^2 + y^2 = 1$ and $0 \leq z \leq 1\}$ in R^3. Let $\omega = z dx$. Fix an orientation on M such that the $e_2 = (0, 1, 0), e_3 = (0, 0, 1)$ give a positive basis of $T_p M$ for $p = (1, 0, 0)$.
Compute $\int_M d\omega$ and $\int_{\partial M} \omega$ and verify that they are equal.

Extra Credit: John Nash's Problem.
Is it true that every closed 1-form on $R^3 \backslash \{(0, 0, 0)\}$ is exact?