(1) Let $f\colon R^3 \to R^3$ be given by $f(x,y,z) = (y\sin(z), xe^z, 1+y^2)$. Let $\omega = zdx \wedge dy$. Compute $df^*(\omega)$ and $f^*(d\omega)$ and verify that they are equal .

(2) Prove that every closed C^∞ 1-form on $\mathbb{R}^2$ is exact.

Hint: Let $\omega = P(x,y)dx + Q(x,y)dy$ with $d\omega = 0$. We want to find a function $F(x,y)$ such that $\omega = dF$, i.e. $P = \frac{\partial F}{\partial x}$ and $Q = \frac{\partial F}{\partial y}$.

Define $F(x,y) = \int_0^x P(x,0)dx + \int_0^y Q(x,y)dy$. Use that $d\omega = 0$ to show that $dF = \omega$.

(3) A subset $X \subset \mathbb{R}^n$ is called path connected if for any points $p, q \in X$ there exists a continuous map $\gamma\colon [0,1] \to X$ such that $\gamma(0) = p, \gamma(1) = q$. Let $U\colon \mathbb{R}^n$ be an open path connected set and $f\colon U \to V$ be a C^1 diffeomorphism onto an open set $V \subset \mathbb{R}^n$.

Prove that $\det[df_x] > 0$ for all $x \in U$ or $\det[df_x] < 0$ for all $x \in U$.

(4) Let $\sigma\colon (0,1)^2 \to R^3$ be given by $\sigma(x,y) = (xy, 2x+y, y^2)$. Let ω be a 2-form on R^3 given by $x_1 dx_2 \wedge dx_3 + x_2^2 dx_1 \wedge dx_3$.

Find $\int_\sigma \omega$.

(5) Let $U \subset \mathbb{R}_n$ be open and $w \subset \Omega^1(U)$ be exact. Let $p, q \in U$ be fixed and let $\gamma\colon [0,1] \to U$ be C^1 such that $\gamma(0) = p, \gamma(1) = q$.

Prove that $\int_\gamma \omega$ is independent of γ.

(6) Let $f\colon \mathbb{R}^k \to \mathbb{R}^n$ be C^∞. Let $x = (x_1, \ldots, x_k)$ denote the general point of $\mathbb{R}^k$ and $y = (y_1, \ldots, y_n)$ denote the general point of $\mathbb{R}^n$. Let $\omega = \phi(y)dy_I$ where $i = (i_1 < i_2 < \ldots < i_k)$. Let $f_I(x) = (f_{i_1}(x), \ldots, f_{i_k}(x))$.

Prove that $f^*(\omega) = \phi(f(x)) \det[df_I(x)] dx_1 \wedge dx_2 \wedge \ldots \wedge dx_k$.

Extra Credit: John Nash's Problem.
Is it true that every closed 1-form on $R^3 \backslash \{(0,0,0)\}$ is exact?