(1) (a) Let $u_1, \ldots, u_{n-1} \in \mathbb{R}^n$. Let A be an orthogonal $n \times n$ matrix with $\det A > 0$.
Prove that $A(u_1) \times \ldots A(u_{n-1}) = A(u_1 \times \ldots \times u_{n-1})$.
Hint: let $w \in \mathbb{R}^n$ be any vector. Show that $\langle u_1, \times \ldots \times u_{n-1}, w\rangle = \langle A(u_1), \times \ldots \times A(u_{n-1}), A(w)\rangle$. On the other hand, observe that since A is orthogonal,

$$\langle u_1, \times \ldots \times u_{n-1}, w\rangle = \langle A(u_1, \times \ldots \times u_{n-1}), A(w)\rangle$$

(b) Let $u_1 = (1, 2, 0, 0), u_2 = (0, 1, 1, 0), u_3 = (0, 1, -1, 1)$. Compute $\mathrm{vol}_3(\mathrm{P}(\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3)$ and $||u_1 \times u_2 \times u_3||$ and verify that they are equal.

(c) Let $u_1, \ldots, u_{n-1} \in \mathbb{R}^n$. Prove that $||u_1 \times \ldots \times u_{n-1}|| = \mathrm{vol}_{n-1}(\mathrm{P}(\mathrm{u}_1, \ldots, \mathrm{u}_{n-1}))$.
Hint: First, use part a) to reduce the situation to the case when $u_1, \ldots, u_{n-1} \in \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$. Then show that both sides of the formula change in the same way under elementary row operations on $u_1, \ldots, u_{n-1}$. use it to further reduce the situation to the case $u_1 = e_1, \ldots u_{n-1} = e_{n-1}$.

(2) Let $U \subset \mathbb{R}^n$ be open and let $f\colon U \to \mathbb{R}$ be C^1 where $n \geq 1$. Consider the graph of f with the parametrization $\alpha(x_1, \ldots, x_n) = (x_1, \ldots, x_n, f(x))$.
Prove that $\mathrm{vol}_\mathrm{n}(\alpha) = \int_\mathrm{U}^{\mathrm{ext}} \sqrt{1 + (\frac{\partial \mathrm{f}}{\partial \mathrm{x}_1})^2 + \ldots + (\frac{\partial \mathrm{f}}{\partial \mathrm{x}_\mathrm{n}})^2}$

(3) Let $f\colon (-1, 1) \to \mathbb{R}$ be given by $f(t) = \sqrt{|t|}$.
Prove that the graph of f is not a manifold.