Solutions to Practice Final Exam 1

1. (12 pts) Give the following definitions

   (a) an open set in $\mathbb{R}^n$.

   (b) a differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$ at a point p.

   (c) an integrable function f on a rectangle $A \subset \mathbb{R}^n$.

   (d) an alternating k-tensor on a vector space V.

   (e) a k-dimensional manifold in $\mathbb{R}^n$.

Solution

(a) A set $U \subset \mathbb{R}^n$ is open if for every point $p \in U$ there exists $\epsilon > 0$ such that $B(p, \epsilon) \subset U$.

(b) A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable at $p \in \mathbb{R}^n$ if there exists a linear map $L\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{f(p+h) - f(p) - L(h)}{|h|} = 0$$

(c) A function $f\colon A \to \mathbb{R}$ is called integrable if f is bounded and

$$\limsup_{P \text{ partition of } A} L(f, P) = \liminf_{P \text{ partition of } A} U(f, P)$$

(d) a k-tensor T on a vector space V is called alternating if for any vectors $v_1, \ldots, v_k \in V$ and any $1 \le i \le j \le k$ we have

$$T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -T(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

(e) A set $M \subset \mathbb{R}^n$ is a k-dimensional C^r-manifold without a boundary if for every point $p \in M$ there exists a set $U \subset M$ which is open in M, an open subset $V \subset \mathbb{R}^k$ and a C^r map $f\colon V \to \mathbb{R}^n$ such that

   i. $f(V) = U$ and $f\colon V \to U$ is 1-1 and onto;

   ii. $rank[df_x] = k$ for any $x \in V$;

   iii. $f^{-1}\colon U \to V$ is continuous.

2. (10 pts) Let A be a rectangle in $\mathbb{R}^n$. Suppose $f, g\colon A \to \mathbb{R}$ are integrable on A.

   Prove that $f + g$ is also integrable on A.

Solution

First observe that if Q is any rectangle then $m_{f+g}Q \geq m_f Q + m_g Q$. Indeed, for any $x \in Q$ we have $f(x)+g(x) \geq f(x)+m_f Q \geq m_f Q + m_g Q$. Since this is true for any $x \in Q$ this implies that $m_{f+g}Q \geq m_f Q + m_g Q$. Therefore, for any partition P of A we have $L(f+g,P) = \sum_{Q\in P} m_{f+g}Q\text{vol}Q \geq \sum_{Q\in P}(m_f Q + m_g Q)Q\text{vol}Q = L(f,P) + L(g,P)$.

Next, note that for any partitions P_1, P_2 of A and any common refinement P of P_1, P_2 we have $L(f,P) \geq L(f,P_1)$ and $L(g,P) \geq L(g,P_2)$. Therefore $\underline{\int}_A f + \underline{\int}_A g = \sup_{P_1} L(f,P_1) + \sup_{P_2} L(f,P_2) \leq \sup_P L(f,P) + L(g,P) \leq \sup_P L(f+g,P) = \underline{\int}_A f+g$.

Similarly, $\overline{\int}_A f + \overline{\int}_A g \geq \overline{\int}_A f+g$. Together with the above this gives

$$\underline{\int}_A f + \underline{\int}_A g \leq \underline{\int}_A f+g \leq \overline{\int}_A f+g \leq \overline{\int}_A f + \overline{\int}_A g$$

Since, $\overline{\int}_A f = \underline{\int}_A f$ and $\overline{\int}_A g = \underline{\int}_A g$. This implies that

$$\underline{\int}_A f+g = \overline{\int}_A f+g = \int_A f + \int_A g \quad \square$$

3. (10 pts) Let A be a subset of $\mathbb{R}^n$. Prove that $A \cup br(A)$ is closed.

Solution

Let $U = \mathbb{R}^n \backslash (A \cup br(A))$. We claim that U is open. Indeed, by definition if $p \in U$ then there exists $\epsilon > 0$ such that $B(p,\epsilon) \cap A = \emptyset$. Then we also have that $B(p,\epsilon) \cap br(A) = \emptyset$, i.e. $B(p,\epsilon \subset U$. Indeed, if not then there exists $q \in B(p,\epsilon) \cap br(A)$. Pick $\delta > 0$ such that $\delta + d(q,p) < \epsilon$. Then $B(q,\delta) \cap A \neq \emptyset$ since $q \in br(A)$. But $B(q,\delta) \subset B(p,\epsilon)$ by the triangle inequality. This means that $B(p,\epsilon) \cap A \neq \emptyset$ also. This is a contradiction. Therefore, $B(p,\epsilon) \subset U$. Since $p \in U$ was arbitrary this implies that U is open. Hence, $A \cup br(A)$ is closed.

4. (8 pts) Let $C \subset \mathbb{R}^n$ be compact. Let $f\colon C \to \mathbb{R}$ be continuous.

   Prove that $f(C)$ is bounded. **You are not allowed to use any theorems about compact sets in the proof.**

Solution

Let $U_n = \{-n < f(x) < n\}$. Then U_n is open in C for any n. Obviously, $\cup_n U_n = C$ and hence it's an open cover of C. By compactness we can choose a finite subcover. Since $U_n \subset U_m$ for $m < n$ this implies that $C = U_n$ for some n. $\square$

5. (10 pts) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = |xy|$.

   Show that f is differentiable at $(0,0)$ and compute $df(0,0)$.

Solution

We claim that $df(0,0) = 0$. To verify that we check the definition

$$\lim_{h\to 0} \frac{f(p+h) - f(p) - L(h)}{|h|} = 0$$

where $h = (h_1, h_2) \in \mathbb{R}^2$. The above limit reduces to

$$\lim_{h\to 0} \frac{|h_1 h_2|}{\sqrt{h_1^2 + h_2^2}} = \lim_{h\to 0} |h_1| \cdot \frac{|h_2|}{\sqrt{h_1^2 + h_2^2}} = 0$$

since $\lim_{h\to 0} |h_1| = 0$ and $\frac{|h_2|}{\sqrt{h_1^2+h_2^2}} \leq 1$ for any $h \neq 0$.

6. (8 pts) Let V be an n-dimensional vector space and $\langle \cdot, \cdot \rangle$ be an inner product on V. Let $e_1, \ldots, e_n$ be an orthonormal basis of V. Recall that we use the following notation. For $I = (i_1, \ldots i_k)$ where $1 \leq i_j \leq n$ denote $e_I^* = e_{i_1}^* \otimes \ldots \otimes e_{i_k}^*$.

   Prove that $\{e_I^*\}_{I=(i_i,\ldots,i_k)}$ are linearly independent.

Solution

Suppose $\sum_I \lambda_i e_I^* = 0$. Let's fix $J = (j_1, \ldots j_k)$ and compute $0 = e_I^*(e_{j_1}, \ldots, e_{j_k}) = \sum_I \lambda_i e_I^*(e_{j_1}, \ldots, e_{j_k}) = \sum_I \lambda_i \delta_{IJ} = \lambda_J$. This means that $\lambda_J = 0$ for any J and hence $\{e_I^*\}_{I=(i_i,\ldots,i_k)}$ are linearly independent. □

7. (10 pts) Let $f = f^1(x,y), f^2(x,y))\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map satisfying

$$f(x,0) = (\cos x, x), f(0,y) = (1+y, \sin y)$$

   Prove that for some open set U containing $(0,0)$ the set $V = f(U)$ is open and $f\colon U \to V$ is a diffeomorphism and compute $d(f^{-1})(1,0)$.

Solution

We compute $D_1 f(0,0) = (-\sin 0, 1) = (0,1)$ and $D_2 f(0,0) = (1, \cos 0) = (1,1)$. Therefore the matrix of partial derivatives $[df(0,0)]$ is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Note that $f(0,0) = (\cos 0, 0) = (1,0)$. Observe that $\det A = -1 \neq 0$ and hence A is invertible. We compute

$$A^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

By Inverse Function Theorem there is an open set U containing $(0,0)$ such that the set $V = f(U)$ is open and $f\colon U \to V$ is a diffeomorphism and

$$[d(f^{-1})(1,0)] = A^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

8. (10 pts) Let $U = \{(x,y) \in \mathbb{R}^2 |$ such that $x > 1, 1 < y < 2\}$. Let $f\colon U \to \mathbb{R}$ be given by $f(x,y) = \frac{1}{xy}$.

   Does $\int_U^{ext} f$ exist? If yes, compute it, if not, explain why not. Give a careful justification of your answer.

Solution

Let $U_n = (1,n) \times (1,2)$. Then U_n is an open exhaustion of U. Observe that $f > 0$ on U. Therefore $\int_U^{ext} f$ exists iff $\lim_{n\to\infty} \int_{U_n^{ext}} f$ exists. Note that f is continuous and bound and hence integrable on every U_n. Therefore $\int_{U_n^{ext}} f$ exists and is equal to $\int_{U_n} f$. Using Fubini we compute $\int_{U_n} f = \int_1^n (\int_1^2 \frac{1}{xy} dy) dx = \int_1^n \frac{\ln 2}{x} dx = \ln 2 \cdot \ln n \to \infty$ as $n \to \infty$. Therefore $\int_U^{ext} f$ does not exist.

9. (12 pts) Let $\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2+y^2+z^2)^{3/2}}$ be a 2-form on $U = \mathbb{R}^3 \backslash (0,0,0)$.

   One can check that $d\omega = 0$. You DO NOT have to verify that.

   (a) Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 |$ such that $x^2 + y^2 + z^2 = 1\}$ with the orientation induced from $B^3 = \{(x,y,z) \in \mathbb{R}^3 |$ such that $x^2 + y^2 + z^2 \le 1\}$.
   Show that $\omega|_{S^2} = dV$

   (b) Show that ω is not exact on U.
   Hint: Assume that $\omega = d\eta$ and use Stokes' formula.

Solution

(a) Note that $\omega|_{S^2} = xdy \wedge dz + ydz \wedge dx + zdx|_{S^2}$ and the unit outward normal filed n on S^2 is given by $N(x,y,z) = (x,y,z)$. Let $p = (x,y,z) \in S^2$. Recall that for any $u, v \in T_pS^2$ we have $dV(u,v) = \langle u \times v, N(p) \rangle = \det A$ where Ais the matrix with rows $u, v, N(p)$.

On the other hand we easily see that $xdy \wedge dz + ydz \wedge dx + zdx(u,v) = \langle u \times v, (x,y,z) \rangle = \det A$ also.

(b) Suppose ω is exact on U. Then $\omega = d\eta$ for some 1-form η. Using Stokes formula we get $\int_{S^2} d\eta = 0$ since $\partial S^2 = \emptyset$. But $d\eta = \omega$. Hence, using a) we get $\int_{S^2} d\eta = \int_{S^2} \omega = \int_{S^2} dV = area(S^2) = 4\pi \neq 0$. This is a contradiction and hence ω is not exact on U. $\square$

10. (10 pts) Let U be the parallelogram with vertices $(0,0),(2,1),(1,3)$ and $(3,4)$. Compute $\int_U x+2y$.

Solution

Consider following change of variables $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$ or $x = 2u+v, y = u+3v$. Then $\det A = 5$ and by the change of variables forumla $\int_U x+2y = \int_{(0,1)^2} 5[(2u+v)+2(u+3v)] = 5\int_{(0,1)^2} 4u + 5\int_{(0,1)^2} 7v = 5\int_0^1(\int_0^1 4udu)dv + 5\int_0^1(\int_0^1 7vdv)du = 5(2+\frac{7}{2}) = \frac{55}{2}$