Past Final Exam

1. (12 pts) Give the following definitions
   (a) an open set in $\mathbb{R}^n$.
   (b) a differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$ at a point p.
   (c) an integrable function f on a rectangle $A \subset \mathbb{R}^n$.
   (d) an alternating k-tensor on a vector space V.
   (e) a k-dimensional manifold in $\mathbb{R}^n$.

2. (10 pts) Let A be a rectangle in $\mathbb{R}^n$. Suppose $f, g\colon A \to \mathbb{R}$ are integrable on A.

   Prove that $f + g$ is also integrable on A.

3. (10 pts) Let A be a subset of $\mathbb{R}^n$. Prove that $A \cup br(A)$ is closed.

4. (8 pts) Let $C \subset R^n$ be compact. Let $f\colon C \to \mathbb{R}$ be continuous.

   Prove that $f(C)$ is bounded. **You are not allowed to use any theorems about compact sets in the proof.**

5. (10 pts) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = |xy|$.

   Show that f is differentiable at $(0, 0)$ and compute $df(0, 0)$.

6. (8 pts) Let V be an n-dimensional vector space and $\langle \cdot, \cdot \rangle$ be an inner product on V. Let $e_1, \ldots, e_n$ be an orthonormal basis of V. Recall that we use the following notation. For $I = (i_1, \ldots i_k)$ where $1 \le i_j \le n$ denote $e_I^* = e_{i_1}^* \otimes \ldots \otimes e_{i_k}^*$.

   Prove that $\{e_I\}_{I=(i_i,\ldots,i_k)}$ are linearly independent.

7. (10 pts) Let $f = f^1(x, y), f^2(x, y))\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map satisfying

   $$f(x, 0) = (\cos x, x), f(0, y) = (1 + y, \sin y)$$

   Prove that for some open set U containing $(0, 0)$ the set $V = f(U)$ is open and $f\colon U \to V$ is a diffeomorphism and compute $d(f^{-1})(1, 0)$.

8. (10 pts) Let $U = \{(x, y) \in \mathbb{R}^2|$ such that $x > 1, 1 < y < 2\}$. Let $f\colon U \to \mathbb{R}$ be given by $f(x, y) = \frac{1}{xy}$.

   Does $\int_U^{ext} f$ exist? If yes, compute it, if not, explain why not. Give a careful justification of your answer.

9. (12 pts) Let $\omega = \frac{xdy\wedge dz + ydz\wedge dx + zdx\wedge dy}{(x^2+y^2+z^2)^{3/2}}$ be a 2-form on $U = \mathbb{R}^3 \backslash (0,0,0)$.

   One can check that $d\omega = 0$. You DO NOT have to verify that.

   (a) Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 |$ such that $x^2+y^2+z^2 = 1\}$ with the orientation induced from $B^3 = \{(x,y,z) \in \mathbb{R}^3 |$ such that $x^2+y^2+z^2 \leq 1\}$.
   Show that $\omega|_{S^2} = dV$

   (b) Show that ω is not exact on U.
   Hint: Assume that $\omega = d\eta$ and use Stokes' formula.

10. (10 pts) Let U be the parallelogram with vertices $(0,0), (2,1), (1,3)$ and $(3,4)$.
    Compute $\int_U x + 2y$.