Also,

$$\langle f_n, f_n\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-n)t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} 1\,dt = 1.$$

In other words, $\langle f_m, f_n\rangle = \delta_{mn}$. ◆

EXERCISES

1. Label the following statements as true or false.
   (a) An inner product is a scalar-valued function on the set of ordered pairs of vectors.
   (b) An inner product space must be over the field of real or complex numbers.
   (c) An inner product is linear in both components.
   (d) There is exactly one inner product on the vector space R^n.
   (e) The triangle inequality only holds in finite-dimensional inner product spaces.
   (f) Only square matrices have a conjugate-transpose.
   (g) If x, y, and z are vectors in an inner product space such that $\langle x, y\rangle = \langle x, z\rangle$, then $y = z$.
   (h) If $\langle x, y\rangle = 0$ for all x in an inner product space, then $y = \mathit{0}$.

2. Let $x = (2, 1+i, i)$ and $y = (2-i, 2, 1+2i)$ be vectors in C^3. Compute $\langle x, y\rangle$, $\|x\|$, $\|y\|$, and $\|x+y\|$. Then verify both the Cauchy–Schwarz inequality and the triangle inequality.

3. In $\mathsf{C}([0,1])$, let $f(t) = t$ and $g(t) = e^t$. Compute $\langle f, g\rangle$ (as defined in Example 3), $\|f\|$, $\|g\|$, and $\|f+g\|$. Then verify both the Cauchy–Schwarz inequality and the triangle inequality.

4. (a) Complete the proof in Example 5 that $\langle\cdot,\cdot\rangle$ is an inner product (the Frobenius inner product) on $\mathsf{M}_{n\times n}(F)$.
   (b) Use the Frobenius inner product to compute $\|A\|$, $\|B\|$, and $\langle A, B\rangle$ for

$$A = \begin{pmatrix} 1 & 2+i \\ 3 & i \end{pmatrix} \quad\text{and}\quad B = \begin{pmatrix} 1+i & 0 \\ i & -i \end{pmatrix}.$$

5. In C^2, show that $\langle x, y\rangle = xAy^*$ is an inner product, where

$$A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}.$$

   Compute $\langle x, y\rangle$ for $x = (1-i, 2+3i)$ and $y = (2+i, 3-2i)$.

6. Complete the proof of Theorem 6.1.

7. Complete the proof of Theorem 6.2.

8. Provide reasons why each of the following is not an inner product on the given vector spaces.
   (a) $\langle (a,b),(c,d)\rangle = ac - bd$ on R^2.
   (b) $\langle A, B\rangle = \operatorname{tr}(A+B)$ on $\mathsf{M}_{2\times 2}(R)$.
   (c) $\langle f(x), g(x)\rangle = \int_0^1 f'(t)g(t)\,dt$ on $\mathsf{P}(R)$, where $'$ denotes differentiation.

9. Let β be a basis for a finite-dimensional inner product space.
   (a) Prove that if $\langle x, z\rangle = 0$ for all $z \in \beta$, then $x = \mathit{0}$.
   (b) Prove that if $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in \beta$, then $x = y$.

10.† Let V be an inner product space, and suppose that x and y are orthogonal vectors in V. Prove that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. Deduce the Pythagorean theorem in R^2.

11. Prove the *parallelogram law* on an inner product space V; that is, show that

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \text{for all } x, y \in \mathsf{V}.$$

   What does this equation state about parallelograms in R^2?

12.† Let $\{v_1, v_2, \ldots, v_k\}$ be an orthogonal set in V, and let $a_1, a_2, \ldots, a_k$ be scalars. Prove that

$$\left\|\sum_{i=1}^{k} a_i v_i\right\|^2 = \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2.$$

13. Suppose that $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are two inner products on a vector space V. Prove that $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_1 + \langle\cdot,\cdot\rangle_2$ is another inner product on V.

14. Let A and B be $n \times n$ matrices, and let c be a scalar. Prove that $(A+cB)^* = A^* + \bar{c}B^*$.

15. (a) Prove that if V is an inner product space, then $|\langle x, y\rangle| = \|x\|\cdot\|y\|$ if and only if one of the vectors x or y is a multiple of the other. *Hint:* If the identity holds and $y \neq \mathit{0}$, let

$$a = \frac{\langle x, y\rangle}{\|y\|^2},$$

and let $z = x - ay$. Prove that y and z are orthogonal and

$$|a| = \frac{\|x\|}{\|y\|}.$$

Then apply Exercise 10 to $\|x\|^2 = \|ay + z\|^2$ to obtain $\|z\| = 0$.

(b) Derive a similar result for the equality $\|x + y\| = \|x\| + \|y\|$, and generalize it to the case of n vectors.

16. (a) Show that the vector space H with $\langle \cdot, \cdot \rangle$ defined on page 332 is an inner product space.

(b) Let $\mathsf{V} = \mathsf{C}([0, 1])$, and define

$$\langle f, g \rangle = \int_0^{1/2} f(t)g(t)\, dt.$$

Is this an inner product on V?

17. Let T be a linear operator on an inner product space V, and suppose that $\|\mathsf{T}(x)\| = \|x\|$ for all x. Prove that T is one-to-one.

18. Let V be a vector space over F, where $F = R$ or $F = C$, and let W be an inner product space over F with inner product $\langle \cdot, \cdot \rangle$. If $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ is linear, prove that $\langle x, y \rangle' = \langle \mathsf{T}(x), \mathsf{T}(y) \rangle$ defines an inner product on V if and only if T is one-to-one.

19. Let V be an inner product space. Prove that

(a) $\|x \pm y\|^2 = \|x\|^2 \pm 2\Re \langle x, y \rangle + \|y\|^2$ for all $x, y \in \mathsf{V}$, where $\Re \langle x, y \rangle$ denotes the real part of the complex number $\langle x, y \rangle$.

(b) $|\,\|x\| - \|y\|\,| \le \|x - y\|$ for all $x, y \in \mathsf{V}$.

20. Let V be an inner product space over F. Prove the *polar identities*: For all $x, y \in \mathsf{V}$,

(a) $\langle x, y \rangle = \frac{1}{4}\|x + y\|^2 - \frac{1}{4}\|x - y\|^2$ if $F = R$;

(b) $\langle x, y \rangle = \frac{1}{4}\sum_{k=1}^{4} i^k \|x + i^k y\|^2$ if $F = C$, where $i^2 = -1$.

21. Let A be an $n \times n$ matrix. Define

$$A_1 = \frac{1}{2}(A + A^*) \quad \text{and} \quad A_2 = \frac{1}{2i}(A - A^*).$$

(a) Prove that $A_1^* = A_1$, $A_2^* = A_2$, and $A = A_1 + iA_2$. Would it be reasonable to define A_1 and A_2 to be the real and imaginary parts, respectively, of the matrix A?

(b) Let A be an $n \times n$ matrix. Prove that the representation in (a) is unique. That is, prove that if $A = B_1 + iB_2$, where $B_1^* = B_1$ and $B_2^* = B_2$, then $B_1 = A_1$ and $B_2 = A_2$.

22. Let V be a real or complex vector space (possibly infinite-dimensional), and let β be a basis for V. For $x, y \in \mathsf{V}$ there exist $v_1, v_2, \ldots, v_n \in \beta$ such that

$$x = \sum_{i=1}^{n} a_i v_i \quad \text{and} \quad y = \sum_{i=1}^{n} b_i v_i.$$

Define

$$\langle x, y \rangle = \sum_{i=1}^{n} a_i \bar{b}_i.$$

(a) Prove that $\langle \cdot, \cdot \rangle$ is an inner product on V and that β is an orthonormal basis for V. Thus every real or complex vector space may be regarded as an inner product space.

(b) Prove that if $\mathsf{V} = \mathsf{R}^n$ or $\mathsf{V} = \mathsf{C}^n$ and β is the standard ordered basis, then the inner product defined above is the standard inner product.

23. Let $\mathsf{V} = \mathsf{F}^n$, and let $A \in \mathsf{M}_{n\times n}(F)$.

(a) Prove that $\langle x, Ay \rangle = \langle A^*x, y \rangle$ for all $x, y \in \mathsf{V}$.

(b) Suppose that for some $B \in \mathsf{M}_{n\times n}(F)$, we have $\langle x, Ay \rangle = \langle Bx, y \rangle$ for all $x, y \in \mathsf{V}$. Prove that $B = A^*$.

(c) Let α be the standard ordered basis for V. For any orthonormal basis β for V, let Q be the $n \times n$ matrix whose columns are the vectors in β. Prove that $Q^* = Q^{-1}$.

(d) Define linear operators T and U on V by $\mathsf{T}(x) = Ax$ and $\mathsf{U}(x) = A^*x$. Show that $[\mathsf{U}]_\beta = [\mathsf{T}]_\beta^*$ for any orthonormal basis β for V.

The following definition is used in Exercises 24–27.

Definition. *Let V be a vector space over F, where F is either R or C. Regardless of whether V is or is not an inner product space, we may still define a norm $\|\cdot\|$ as a real-valued function on V satisfying the following three conditions for all $x, y \in \mathsf{V}$ and $a \in F$:*

(1) *$\|x\| \ge 0$, and $\|x\| = 0$ if and only if $x = 0$.*

(2) *$\|ax\| = |a| \cdot \|x\|$.*

(3) *$\|x + y\| \le \|x\| + \|y\|$.*

24. Prove that the following are norms on the given vector spaces V.

(a) $\mathsf{V} = \mathsf{M}_{m\times n}(F)$; $\quad \|A\| = \max_{i,j} |A_{ij}|$ for all $A \in \mathsf{V}$

(b) $\mathsf{V} = \mathsf{C}([0, 1])$; $\quad \|f\| = \max_{t\in[0,1]} |f(t)|$ for all $f \in \mathsf{V}$

(c) $V = C([0,1])$; $\|f\| = \int_0^1 |f(t)|\,dt$ for all $f \in V$
(d) $V = R^2$; $\|(a,b)\| = \max\{|a|, |b|\}$ for all $(a,b) \in V$

25. Use Exercise 20 to show that there is no inner product $\langle \cdot, \cdot \rangle$ on R^2 such that $\|x\|^2 = \langle x, x \rangle$ for all $x \in R^2$ if the norm is defined as in Exercise 24(d).

26. Let $\|\cdot\|$ be a norm on a vector space V, and define, for each ordered pair of vectors, the scalar $d(x,y) = \|x - y\|$, called the **distance** between x and y. Prove the following results for all $x, y, z \in V$.

(a) $d(x,y) \geq 0$.
(b) $d(x,y) = d(y,x)$.
(c) $d(x,y) \leq d(x,z) + d(z,y)$.
(d) $d(x,x) = 0$.
(e) $d(x,y) \neq 0$ if $x \neq y$.

27. Let $\|\cdot\|$ be a norm on a real vector space V satisfying the parallelogram law given in Exercise 11. Define

$$\langle x, y \rangle = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2\right].$$

Prove that $\langle \cdot, \cdot \rangle$ defines an inner product on V such that $\|x\|^2 = \langle x, x \rangle$ for all $x \in V$.

Hints:

(a) Prove $\langle x, 2y \rangle = 2\langle x, y \rangle$ for all $x, y \in V$.
(b) Prove $\langle x+u, y \rangle = \langle x, y \rangle + \langle u, y \rangle$ for all $x, u, y \in V$.
(c) Prove $\langle nx, y \rangle = n\langle x, y \rangle$ for every positive integer n and every $x, y \in V$.
(d) Prove $m\left\langle \frac{1}{m}x, y \right\rangle = \langle x, y \rangle$ for every positive integer m and every $x, y \in V$.
(e) Prove $\langle rx, y \rangle = r\langle x, y \rangle$ for every rational number r and every $x, y \in V$.
(f) Prove $|\langle x, y \rangle| \leq \|x\|\|y\|$ for every $x, y \in V$. *Hint:* Condition (3) in the definition of norm can be helpful.
(g) Prove that for every $c \in R$, every rational number r, and every $x, y \in V$,

$$|c\langle x, y\rangle - \langle cx, y\rangle| = |(c-r)\langle x, y\rangle - \langle (c-r)x, y\rangle| \leq 2|c-r|\|x\|\|y\|.$$

(h) Use the fact that for any $c \in R$, $|c - r|$ can be made arbitrarily small, where r varies over the set of rational numbers, to establish item (b) of the definition of *inner product*.

28. Let V be a complex inner product space with an inner product $\langle \cdot, \cdot \rangle$. Let $[\cdot, \cdot]$ be the real-valued function such that $[x, y]$ is the real part of the complex number $\langle x, y \rangle$ for all $x, y \in V$. Prove that $[\cdot, \cdot]$ is an inner product for V, where V is regarded as a vector space over R. Prove, furthermore, that $[x, ix] = 0$ for all $x \in V$.

29. Let V be a vector space over C, and suppose that $[\cdot, \cdot]$ is a real inner product on V, where V is regarded as a vector space over R, such that $[x, ix] = 0$ for all $x \in V$. Let $\langle \cdot, \cdot \rangle$ be the complex-valued function defined by

$$\langle x, y \rangle = [x, y] + i[x, iy] \quad \text{for } x, y \in V.$$

Prove that $\langle \cdot, \cdot \rangle$ is a complex inner product on V.

30. Let $\|\cdot\|$ be a norm (as defined in Exercise 24) on a complex vector space V satisfying the parallelogram law given in Exercise 11. Prove that there is an inner product $\langle \cdot, \cdot \rangle$ on V such that $\|x\|^2 = \langle x, x \rangle$ for all $x \in V$.

Hint: Apply Exercise 27 to V regarded as a vector space over R. Then apply Exercise 29.

6.2 THE GRAM–SCHMIDT ORTHOGONALIZATION PROCESS AND ORTHOGONAL COMPLEMENTS

In previous chapters, we have seen the special role of the standard ordered bases for C^n and R^n. The special properties of these bases stem from the fact that the basis vectors form an orthonormal set. Just as bases are the building blocks of vector spaces, bases that are also orthonormal sets are the building blocks of inner product spaces. We now name such bases.

Definition. *Let V be an inner product space. A subset of V is an **orthonormal basis** for V if it is an ordered basis that is orthonormal.*

Example 1

The standard ordered basis for F^n is an orthonormal basis for F^n. ◆

Example 2

The set

$$\left\{\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)\right\}$$

is an orthonormal basis for R^2. ◆