Math 247S **Solutions to the Term Test** **March 5, 2012**

(1) (10 pts) An $n \times n$ matrix A is called skew-symmetric if $A^t = -A$.
Suppose A is an $n \times n$ a real skew-symmetric matrix where n is odd.
Prove that $\text{rank}(A) < n$.
Hint: Consider $\det(A)$.

Solution

Observe that for any $n \times n$ matrix B we have $\det(-B) = (-1)^n \det B$ since det is linear in every row of B and B has n rows. Therefore, for a skew-symmetric matrix A we have $\det A = \det A^t = (-1)^n \det A = -\det A$ if n is odd. Therefore, $\det A = 0$ and hence $\text{rank}(A) < n$.

(2) (10 pts) Which of the following define an inner product? Justify your answer.
a) $\langle A, B\rangle = tr(\underline{AB^t})$ on $V = M_{n\times n}(\mathbb{C})$;
b) $\langle f, g\rangle = f(0)\overline{g(0)}$ on $V = C[0, 1]$;

Solution

a) **False**.Take $A = B = \lambda Id$ where λ is a complex number. Then $\langle A, A\rangle = n\lambda^2$ which is not always a real number. For example take $\lambda = 1 + i$.
b) **False.** Take $f(x) = g(x) = x$. Then $\langle f, f\rangle = f(0)^2 = 0$ but $f \neq 0$.

(3) (10 its) Let $A\colon \mathbb{C}^2 \to \mathbb{C}^2$ be given by

$$A = \begin{pmatrix} 1 & 2i \\ i & -2 \end{pmatrix}$$

Find an orthogonal basis of the image of A^*.

Solution

We have

$$A^* = \bar{A}^t = \begin{pmatrix} 1 & -i \\ -2i & -2 \end{pmatrix}$$

Observe that this matrix has rank=1 since its determinant is equal to $2+2i^2 = 0$. Therefore the image of A^* has dimension 1 and its basis can be given by $A^*(e_1) = \begin{pmatrix} 1 \\ -2i \end{pmatrix}$. This basis is automatically orthogonal as it consists of a single vector.

(4) (10 its) Find the value of k which satisfies the following equation

$$\det\begin{pmatrix} 7c_1 & 7c_2 & 7c_3 \\ 2a_1+3c_1 & 2a_2+3c_2 & 2a_3+3c_3 \\ 3b_1+5a_1 & 3b_2+5a_2 & 3b_3+5a_3 \end{pmatrix} = k\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

Solution

We have

$$\det\begin{pmatrix} 7c_1 & 7c_2 & 7c_3 \\ 2a_1+3c_1 & 2a_2+3c_2 & 2a_3+3c_3 \\ 3b_1+5a_1 & 3b_2+5a_2 & 3b_3+5a_3 \end{pmatrix} = 7\det\begin{pmatrix} c_1 & c_2 & c_3 \\ 2a_1+3c_1 & 2a_2+3c_2 & 2a_3+3c_3 \\ 3b_1+5a_1 & 3b_2+5a_2 & 3b_3+5a_3 \end{pmatrix} =$$

$$7\det\begin{pmatrix} c_1 & c_2 & c_3 \\ 2a_1 & 2a_2 & 2a_3 \\ 3b_1+5a_1 & 3b_2+5a_2 & 3b_3+5a_3 \end{pmatrix} = 14\det\begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ 3b_1+5a_1 & 3b_2+5a_2 & 3b_3+5a_3 \end{pmatrix}$$

$$= 14\det\begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ 3b_1 & 3b_2 & 3b_3 \end{pmatrix} = 42\det\begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = -42\det\begin{pmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

$$42\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

(5) (10 pts) Let $V = \mathbb{R}^2$ with the following inner product $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2$.

Find an orthogonal basis of V.

Solution

Let $v_1 = (1, 0), v_2 = (0, 1)$. It's a basis of V. We will apply Gramm-Schmidt orthogonalization to transform this basis into an orthogonal basis.

We set $w_1 = v_1 = (1, 0)$. Then we take $w_2 = v_2 - \frac{\langle v_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$.

We compute $\langle v_2, v_1 \rangle = \langle (0, 1), (1, 0) \rangle = 1$. Also $\langle v_1, v_1 \rangle = \langle (1, 0), (1, 0) \rangle = 1$. hence $w_2 = v_2 - v_1 = (1, -1)$.

Thus $w_1 = (1, 0), w_2 = (-1, 1)$ is an orthogonal basis of V.