Math 247S **Solutions to Practice Term Test**
Winter 2012

(1) Let V be a complex vector space with two inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$.

Suppose $\langle v,v\rangle_1 = \langle v,v\rangle_2$ for any $v \in V$.

Prove that $\langle u,v\rangle_1 = \langle u,v\rangle_2$ for any $u, v \in V$.

Solution

First observe that for any inner product $\langle v,v\rangle$ we have $\langle u+v,u+v\rangle = \langle u,u\rangle + \langle v,v\rangle + \langle u,v\rangle + \langle v,u\rangle = \langle u,u\rangle + \langle v,v\rangle + \langle u,v\rangle + \overline{\langle u,v\rangle} = \langle u,u\rangle + \langle v,v\rangle + 2\mathrm{Re}\langle u,v\rangle$.

Therefore $\mathrm{Re}\langle u,v\rangle_1 = \mathrm{Re}\langle u,v\rangle_2$ for any $u, v \in V$. Applying this to iu, v we get $\mathrm{Re}\langle iu,v\rangle_1 = \mathrm{Re}\langle iu,v\rangle_2$ for any $u, v \in V$. However, $\mathrm{Re}\langle iu,v\rangle_j = \mathrm{Re}(i\langle u,v\rangle_j) = -\mathrm{Im}\langle u,v\rangle_j$ for $j = 1, 2$ and hence $\mathrm{Im}\langle u,v\rangle_1 = \mathrm{Im}\langle u,v\rangle_2$ for any $u, v \in V$. Combining the above we get $\langle u,v\rangle_1 = \mathrm{Re}\langle u,v\rangle_1 + i\mathrm{Im}\langle u,v\rangle_1 = \mathrm{Re}\langle u,v\rangle_2 + i\mathrm{Im}\langle u,v\rangle_2 = \langle u,v\rangle_2$ for any $u, v \in V$.

(2) For which real values of a, b, c is the matrix the matrix

$$A = \begin{pmatrix} a & b & -c \\ -b & a & 0 \\ ac & 0 & 1 \end{pmatrix}$$

invertible? Find the formula for A^{-1} for those values of (a, b, c) for which A^{-1} exists.

Solution

First we compute $\det A$ by expanding along the last row:

$$\det A = ac(0 + ac) - 0 + 1 \cdot (a^2 + b^2) = a^2c^2 + a^2 + b^2$$

We see that $\det A \geq 0$ and it's equal to zero iff $a = b = 0$ and c is arbitrary. Thus A is invertible iff $(a, b) \neq (0, 0)$.

When $(a,b) \neq (0,0)$ we compute A^{-1} by the general formula for the inverse matrix

$$A^{-1} = \frac{1}{a^2c^2 + a^2 + b^2} \begin{pmatrix} a & -b & ac \\ b & a + ac^2 & bc \\ -a^2c & abc & a^2 + b^2 \end{pmatrix}$$

(3) Let $V = \mathbb{R}^\infty$ i.e., V is the space of infinite sequences of real numbers $a = (a_1, a_2, \ldots)$ where all but finitely many a_i are zero for every $a \in V$.

Let $f\colon V \to \mathbb{R}$ be given by $f(a) = \sum_{i=1}^{\infty} a_i$.

Is it true that there exists $v \in V$ such that $f(a) = \langle a, v\rangle$ for all $a \in V$? if yes, find v. If not, explain why not.

Solution

Suppose such $v = (v_1, v_2, \ldots)$ exists. Let $e_i = (0, 0, \ldots, 1, 0 \ldots)$ be the i-th coordinate vector. Then $1 = f(e_i) = \langle e_i, v\rangle = v_i$. this $v_i = 1$ for every i. However, by definition of $\mathbb{R}^\infty$ we must have that $v_i = 0$ for all sufficiently large i. This is a contradiction and therefore no such v exists.

(4) Mark true or false. If true, give an argument why, if false, give a counterexample.

a) Let V be a finite dimensional complex vector space with inner product and let $f\colon V \to \mathbb{C}$ be a linear map. Then there exists $v \in V$ such that $f(u) = \langle v, u\rangle$ for every $u \in V$.

False. As defined the function $g(u) = \langle v, u\rangle$ is complex anti-linear rather than complex linear since $g(\lambda u) = \langle v, \lambda u\rangle = \bar{\lambda}\langle v, u\rangle = \bar{\lambda} g(u)$.

b) Similar matrices have equal determinants.

True. If $A = T^{-1}BT$ then $\det A = \det(T^{-1}) \cdot \det B \cdot \det T = (\det T)^{-1} \cdot \det B \cdot \det T = \det B$

c) Every orthonormal set of vectors is linearly independent.

True. Let $v_1, \dots$ be orthonormal. Suppose $\sum_{i=1}^n \lambda_i v_i = 0$. Fix some $j \le n$ then we have

$$0 = \langle \sum_{i=1}^n \lambda_i v_i, v_j \rangle = \sum_{i=1}^n \lambda_i \langle v_i, v_j \rangle = \sum_{i=1}^n \lambda_i \delta_{ij} = \lambda_j$$

Since $j \le n$ was arbitrary we conclude that $v_1, \dots$ are linearly independent.

d) Let V be a finite-dimensional vector space with inner product. Then for any $S \subset V$ we have $(S^\perp)^\perp = S$.

False. Let $V = \mathbb{R}$ with the standard inner product and $S = [0, 1]$. Then $S^\perp = 0$ and $(S^\perp)^\perp = \mathbb{R} \ne S$.

(5) Let $W = \{(x, y, z) \in \mathbb{R}^3$ such that $x + 2y - z = 0\}$.

a) Find an orthogonal basis of W;

b) Find the orthogonal projection of $(1, 1, 2)$ to W.

Solution

a) First we find that $v_1 = (1, 0, 1), v_2 = (0, 1, 2)$ is a basis of W. Next we apply Gramm-Schmidt orthogonalization to it.

$w_1 = v_1, w_2 = v_2 - \frac{\langle v_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = v_2 - \frac{2}{2} v_1 = v_2 - v_1 = (-1, 1, 1)$. Thus $w_1 = (1, 0, 1), w_2 = (-1, 1, 1)$ is an orthogonal basis of W.

b) Let $v = (1, 1, 2)$. Let $P \colon V \to W$ be the orthogonal projection map. Then $P(v) = \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle v, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 = \frac{3}{2} w_1 + \frac{2}{3} w_2 = \frac{3}{2}(1, 0, 1) + \frac{2}{3}(-1, 1, 1) = (\frac{5}{6}, \frac{2}{3}, \frac{11}{6})$

(6) Let $v_1, \dots v_n$ be vectors in $\mathbb{R}^n$. Let G be an $n \times n$ matrix with $G_{ij} = \langle v_i, v_j \rangle$. Let P be the parallelepiped spanned by $v_1, \dots v_n$.

Prove that $\mathrm{vol(P)} = \sqrt{\mathrm{det\, G}}$.

Hint: Look at the matrix A with rows $v_1, \dots v_n$.

Solution

Let A be the matrix with rows $v_1, \dots v_n$. Then we have that $AA^t = G$. Therefore, $\det G = \det A \cdot \det A^t = (\det A)^2 = (\mathrm{vol}(P))^2$ since $|\det A| = \mathrm{vol}(P)$.