Math 247S Practice Term Test Winter 2012

Rules: No books, no notes. You have 50 minutes to complete the test. Note: the actual test is shorter than the practice test.

(1) Let V be a complex vector space with two inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$.

Suppose $\langle v, v \rangle_1 = \langle v, v \rangle_2$ for any $v \in V$.

Prove that $\langle u, v \rangle_1 = \langle u, v \rangle_2$ for any $u, v \in V$.

(2) For which real values of a, b, c is the matrix the matrix

$$A = \begin{pmatrix} a & b & -c \\ -b & a & 0 \\ ac & 0 & 1 \end{pmatrix}$$

invertible? Find the formula for A^{-1} for those values of (a, b, c) for which A^{-1} exists.

(3) Let $V = \mathbb{R}^\infty$ i.e., V is the space of infinite sequences of real numbers $a = (a_1, a_2, \ldots)$ where all but finitely many a_i are zero for every $a \in V$.

Let $f\colon V \to \mathbb{R}$ be given by $f(a) = \sum_{i=1}^\infty a_i$.

Is it true that there exists $v \in V$ such that $f(a) = \langle a, v \rangle$ for all $a \in V$? if yes, find v. If not, explain why not.

(4) Mark true or false. If true, give an argument why, if false, give a counterexample.

a) Let V be a finite dimensional complex vector space with inner product and let $f\colon V \to \mathbb{C}$ be a linear map. Then there exists $v \in V$ such that $f(u) = \langle v, u \rangle$ for every $u \in V$.

b) Similar matrices have equal determinants.

c) Every orthonormal set of vectors is linearly independent;

d) Let V be a finite-dimensional vector space with inner product. Then for any $S \subset V$ we have $(S^\perp)^\perp = S$

(5) Let $W = \{(x, y, z) \in \mathbb{R}^3$ such that $x + 2y - z = 0\}$.
a) Find an orthogonal basis of W;
b) Find the orthogonal projection of $(1, 1, 2)$ to W.

(6) Let $v_1, \ldots v_n$ be vectors in $\mathbb{R}^n$. Let G be an $n \times n$ matrix with $G_{ij} = \langle v_i, v_j \rangle$. Let P be the parallelepiped spanned by $v_1, \ldots v_n$.

Prove that $\mathrm{vol(P)} = \sqrt{\mathrm{det\,G}}$.

Hint: Look at the matrix A with rows $v_1, \ldots v_n$.