Solutions to Practice Final

(1) Recall that for a group G its center $Z(G)$ is defined to be the set of all elements $h \in G$ such that $hg = gh$ for any $g \in G$.

(a) Prove that $Z(G)$ is a normal subgroup of G.

(b) Show that $Z(S_n)$ is trivial for any $n > 2$.

(c) Find the center of $GL(n, \mathbb{R})$.

Solution

(a) Let $g \in G$ and $h \in Z(G)$ be any elements. Then $ghg^{-1} = hgg^{-1} = h \in Z(G)$ and hence $Z(G) \trianglelefteq G$ is normal in G.

(b) First observe that for any $\sigma \in S_n$ and any $i \neq j$ we have $\sigma \circ (ij) \circ \sigma^{-1} = (\sigma(i)\sigma(j))$. Suppose $\sigma \in Z(G)$. Then by above $(\sigma(i)\sigma(j)) = \sigma \circ (ij) \circ \sigma^{-1} = (ij)$. Therefore $\sigma(i) = i, \sigma(j) = j$ or $\sigma(i) = j, \sigma(j) = i$. Suppose $\sigma \neq e$ and $n \geq 3$. Then there is $1 \leq i \leq n$ such that $j = \sigma(i) \neq i$. Let $1 \leq k \leq n$ be different from both i and j (this is where we need $n \geq 3$). Then $\sigma \circ (ik) \circ \sigma^{-1} = (\sigma(i)\sigma(k)) = (j\sigma(k)) \neq (ik)$ since $j \neq i$ and $j \neq k$.

(c) We claim that $Z(GL(n, \mathbb{R})) = \{\lambda Id | \lambda \in \mathbb{R} \backslash \{0\}\}$.
Suppose $A \in Z(GL(n, \mathbb{R}))$. Let $B \in GL(n, \mathbb{R})$ be the diagonal matrix with $b_{ii} = \lambda_i$ where $\lambda_i \neq 0$ for any i and $\lambda_i \neq \lambda_j$ for any $i \neq j$ (for example we can take $\lambda_i = i$). Then we must have $AB = BA$. We compute that $(AB)_{ij} = a_{ij}\lambda_j$ and $(BA)_{ij} = a_{ij}\lambda_i$. For any $i \neq j$ this implies $a_{ij}\lambda_j = a_{ij}\lambda_i, a_{ij}(\lambda_i - \lambda_j) = 0, a_{ij} = 0$. Hence A is diagonal.
Suppose some of the diagonal entries of A are different. WLOG, say, $a_{11} \neq a_{22}$. Take B to be the permutation matrix such that $B(e_1) = e_2, B(e_2) = e_1$ and $B(e_i) = e_i$ for $i > 2$. Then $C = BAB^{-1}$ is a diagonal matrix with $c_{11} = a_{22}$ and $c_{22} = a_{11}$. Hence $C \neq A$ which means that A can not be in the centre of G. Therefore all diagonal entries of A are equal i.e. $A = \lambda$ Id for some $\lambda \neq 0$. It's obvious that any such matrix commutes with any $n \times n$ matrix and therefore $Z(GL(n, \mathbb{R})) = \{\lambda Id | \lambda \in \mathbb{R} \backslash \{0\}\}$.

(2) Let V be the space of quadratic polynomials with real coefficients with the inner product given by

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt$$

Find an orthogonal basis of V.

Solution

Let's choose a basis of V. The most natural basis to take is $v_1 = 1, v_2 = t, v_3 = t^2$. We orthogonolize it using Gramm-Schmidt. Set $w_1 = v_1 = 1, w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$. We compute $\langle w_1, w_1 \rangle = \int_0^1 1 = 1$ and $\langle v_2, w_1 \rangle = \int_0^1 t dt = \frac{t^2}{2}|_0^1 = \frac{1}{2}$. Therefore, $w_2 = v_2 - \frac{1}{2}w_1 = t - \frac{1}{2}$.

Next we find $w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$. We compute $\langle v_3, w_1 \rangle = \int_0^1 t^2 dt = \frac{1}{3}$, $\langle w_2, w_2 \rangle = \int_0^1 (t - \frac{1}{2})^2 dt = \frac{(t-\frac{1}{2})^3}{3}|_0^1 = \frac{1}{12}$ and $\langle v_3, w_2 \rangle = \int_0^1 t^2(t - \frac{1}{2})dt = \int_0^1 t^3 - \frac{1}{2}t^2 dt = \frac{t^4}{4} - \frac{t^3}{6}|_0^1 = \frac{1}{12}$. Therefore, $w_3 = v_3 - \frac{1}{3}w_1 - w_2 = t^2 - \frac{1}{3} - (t - \frac{1}{2}) = t^2 - t + \frac{1}{6}$

(3) Give the definitions of the following notions:
a) A normal linear map;
b) A self-adjoint matrix;
c) An inner product on a vector space over $\mathbb{C}$.

Solution

a) Let V be a vector space with an inner product. A linear map $T\colon V \to V$ is called normal if $AA^* = A^*A$.
b) An $n \times n$ matrix is called self-adjoint if $A = \bar{A}^t$.
c) A inner product on a vector space V over $\mathbb{C}$ is a map $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{C}$ which satisfies the following properties
(i) $\langle \lambda_1 u_1 + \lambda_2 u_2, v\rangle = \lambda_1 \langle u_1, v\rangle + \lambda_2 \langle u_2, v\rangle$ for any $u_1, u_2, v \in V, \lambda_1, \lambda_2 \in \mathbb{C}$.
(ii) $\langle u, v\rangle = \overline{\langle v, u\rangle}$ for any $u.v \in V$.
(iii) $\langle u, u\rangle \geq 0$ for any $u \in V$ and $\langle u, u\rangle = 0$ if and only if $u = 0$.

(4) Which of the following define an inner product?
a) $\langle A, B\rangle = tr(A + B)$ on $M_{2\times 2}(R)$;
b) $\langle f, g\rangle = \int_0^{1/2} f(t)\bar{g}(t)dt$ on $C[0, 1]$;
c) $\langle f, g\rangle = \int_0^1 f(t)\bar{g}(t)w_1(t)dt + \int_0^1 f'(t)\bar{g}'(t)w_2(t)dt$ on $C^\infty[0, 1]$ (the space functions on $[0, 1]$ having derivatives of all orders) where $w_1(t) > 0, w_2(t) > 0$ are continuous positive functions on $[0, 1]$.

Solution

a) Not a scalar product because $\langle -Id, -Id\rangle = -2n < 0$.
b) Not a scalar product. Take

$$f(x) = \begin{cases} 0 \text{ if } x \leq 1/2 \\ x - \frac{1}{2} \text{ if } x > \frac{1}{2} \end{cases}$$

Then f is continuous and not identically zero on $[0, 1]$ but $\langle f, f\rangle = \int_0^{1/2} f^2(t)dt = 0$.
c) A scalar product. It's easy to check that all the properties of a scalar product are satisfied.

(5) Mark true or false. If true, give an argument why, if false, give a counterexample. Let V be a finite dimensional vector space.
a) If $T\colon V \to V$ satisfies $|Tv| = |v|$ for any $v \in V$ then T is normal.
b) The adjoint of a normal operator on a finite dimensional vector space is normal.
c) If $T\colon V \to V$ is a self-adjoint linear operator satisfying $\langle Tx, x\rangle > 0$ for any $x \neq 0$ then $\langle x, y\rangle_2 = \langle Tx, y\rangle$ is an inner product on V.
d) Every orthonormal set of vectors is linearly independent;
e) If $T\colon V \to V$ satisfies $|\lambda| = 1$ for any eigenvalue λ of T then T is unitary.

Solution

a) **True.** If T satisfies $|Tv| = |v|$ for any $v \in V$ then T is unitary and hence normal.
b) **True.** Since V is finite dimensional we have that $(T^*)^* = T$. Hence $(T^*)^*T^* = TT^* = T^*T = T^*(T^*)^*$ and therefore T^* is normal.
c) **True.** It's easy to check that all the properties of a scalar product are satisfied.
d) **True.** Suppose $v_1, \ldots v_k$ are orthonormal and $\sum_{i=1}^k \lambda_i v_i = 0$. Fix any $1 \leq j \leq k$ then $0 = \langle \sum_{i=1}^k \lambda_i v_i, v_j\rangle = \sum_{i=1}^k \lambda_i \langle v_j, v_i\rangle = \lambda_j$. Thus $\lambda_j = 0$ for every j.

e) **False.** For example take $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then all eigenvalues of A are equal to 1 but A is not unitary.

(6) Let $T\colon V \to V$ be a normal operator satisfying $T^2 = I$. Show that there exists a subspace $W \subset V$ such that $T(x) = x$ for any $x \in W$ and $T(x) = -x$ for any $x \in W^{\perp}$.

Solution

Since $T^2 = Id$ any eigenvalue λ of T satisfies $\lambda^2 = 1$. Therefore $\lambda = \pm 1$. Now the result follows from the Spectral decomposition theorem with W ($W^{\perp}$) equal to the $+1$ (respectively -1) eigenspace of T.

(7) Let A be a real $n \times n$ matrix all of whose eigenvalues are real. Show that A is normal if and only if it is symmetric.

Solution

If A is symmetric it is obviously normal. Now suppose A is normal and all its eigenvalues are real. Then viewed as a complex matrix it's unitarily equivalent to a diagonal matrix, i.e. $A = UDU^*$ where U is unitary and D is diagonal with real entries. Then $A^* = (UDU^*)^* = (U^*)^* D^* U^* = UDU^* = A$ which means that A is symmetric.

(8) Let $A = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}$. Write A as $QDQ^{\star}$ where D is diagonal and Q is unitary.

Solution

We first find the eigenvalues of A. We have $P_A(\lambda) = (\lambda - 2)^2 - (-i^2) = (\lambda - 2)^2 - 1^2 = (\lambda - 1)(\lambda - 3)$. Hence the eigenvalues are $\lambda_1 = 1, \lambda_2 = 3$.

Next we find the corresponding eigenvectors. For $\lambda_1 = 1$ we get that an eingenvector $v_1 = \begin{pmatrix} x \\ y \end{pmatrix}$ satisfies $(A - Id)v_1 = 0$ or

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad x + iy = 0$$

We can take $v_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$. Similarly, we solve $(A - 3Id)v_2 = 0$ and find $v_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$. Then v_1 and v_2 are orthogonal and we can make them orthonormal by rescaling. Set $\tilde{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}$ and $\tilde{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$. Then U with the columns $\tilde{v}_1, \tilde{v}_2$ (i.e. $U = \begin{pmatrix} \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$) is unitary and satisfies $A = UDU^*$ where $D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

(9) Let $H \leq G$ be a subgroup of index $[G : H] = 2$.

Prove that H is a normal subgroup of G.

Solution

Pick $g_0 \in G \backslash H$. Since $[G : H] = 2$ we have that G consists of exactly two cosets $eH = H$ and $g_0 H$. If $g \in H$ and $h \in H$ then, obviously, $ghg^{-1} \in H$ also since H is a subgroup. If $g \notin H$ then $g = g_0 h_0$ for some $h_0 \in H$. Suppose for some $h \in H$ we have $ghg^{-1} \notin H$. Then $ghg^{-1} = g_0 h_1$ for some $h_1 \in H$. This gives $g_0 h_1 = ghg^{-1} = (g_0 h_0) h (g_0 h_0)^{-1} = g_0 h_0 h h_0^{-1} g_0^{-1}$. Multiplying both sides by g_0^{-1} on the left we get $h_1 = (h_0hh_0^{-1})g_0^{-1}, h_1g_0 = h_0hh_0^{-1}, g_0 = h_1^{-1}h_0hh_0^{-1} \in H$. This is a contradiction and hence $ghg^{-1} \in H$ which means that $H \trianglelefteq G$ is normal in G.

(10) Let $\sigma \in S_n$ be a permutation. Define an $n \times n$ matrix P_σ by the formula $(P_\sigma)_{ij} = \delta_{i\sigma(j)}$. Prove that $\sigma \mapsto P_\sigma$ is a homomorphism and $\det P_\sigma = \text{sign}(\sigma)$.

Solution

By construction we have that $P_\sigma(e_i) = e_{\sigma(i)}$ where $e_1, \dots e_n$ is the standard basis of $\mathbb{R}^n$. Therefore $P_{\sigma\tau}(e_i) = e_{\sigma\tau(i)} = P_\sigma(e_{\tau(i)}) = P_\sigma(P_\tau(e_i))$ for any i which means that $P_{\sigma\tau} = P_\sigma P_\tau$ and hence $\sigma \mapsto P_\sigma$ is a homomorphism and $\phi\colon S_n \to \mathbb{R}\backslash\{0\}$ given by $\phi(\sigma) = \det P_\sigma$ is also a homomorphism as a composition of two homomorphisms. For any transposition σ we have $\phi(\sigma) = \det P_\sigma = -1 = \text{sign}(\sigma)$. Thus ϕ and sign are homomorphisms that agree all transpositions. Since transpositions generate S_n this means that $\phi(\sigma) =$sign(σ) for any $\sigma \in S_n$.

(11) Let G be a group. For any $a, b \in G$ *the commutator* of a and b is defined as $[a, b] = aba^{-1}b^{-1}$. Further, denote by $[G, G]$ the subgroup of G generated by the commutators $[a, b]$ where $a, b \in G$ are arbitrary.

Prove that $[G, G] \trianglelefteq G$ is a normal subgroup of G.

Solution

First observe that $[a, b]^{-1} = [b, a]$. Therefore $[G, G]$ is equal to the set of elements of the form $h_1 \cdot h_2 \cdot \ldots \cdot h_k$ where every h_i has the form $h_i = [a_i, b_i]$ for some $a_i, b_i \in G$. Next observe that $g[a, b]g^{-1} = [gag^{-1}, gbg^{-1}] \in [G, G]$ for any $a, b, g \in G$. Therefore for any $g \in G$ and $h \in [G, G]$ we can write h as $h_1 \cdot h_2 \cdot \ldots \cdot h_k$ with $h_i = [a_i, b_i]$. Then $ghg^{-1} = (gh_1g^{-1})(gh_2g^{-1}) \ldots (gh_kg^{-1}) \in [G, G]$ by the observation above.

(12) Let $T\colon V \to V$ be a skew-adjoint linear operator. Prove that all the eigenvalues of T are purely imaginary (i.e every eigenvalue λ has the form $\lambda = ia$ where a is real).

Solution

Let $Tv = \lambda v$ where $v \neq 0$. By rescaling we can assume $|v| = 1$. Then $\lambda = \lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle Tv, v\rangle = \langle v, T^*v\rangle = \langle v, (-T)v\rangle = \langle v, -\lambda v\rangle = -\bar{\lambda}\langle v, v\rangle = -\bar{\lambda}$. Thus $\lambda = -\bar{\lambda}$ and hence λ is purely imaginary.

(13) Let $f\colon M_{n\times n}(\mathbb{C}) \to \mathbb{C}$ be a function satisfying the following conditions
  (a) $f(Id) = 1$
  (b) $f(A') = -f(A)$ if A is obtained by interchanging two rows of A.
  (c) $f(A') = f(A)$ if A' is obtained from A by adding a multiple of a row to another row.

Is it true that $f(A) = \det A$? If true give a proof, if false, give a counterexample.

Solution

This is false. For example, $f(A) = (\det A)^3$ satisfies all these conditions.