Practice Final

(1) Recall that for a group G its center $Z(G)$ is defined to be the set of all elements $h \in G$ such that $hg = gh$ for any $g \in G$.
  (a) Prove that $Z(G)$ is a normal subgroup of G.
  (b) Show that $Z(S_n)$ is trivial for any $n > 2$.
  (c) Find the center of $GL(n, \mathbb{R})$.

(2) Let V be the space of quadratic polynomials with real coefficients with the inner product given by

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt$$

Find an orthogonal basis of V.

(3) Give definitions of the following notions:
  a) A normal linear map;
  b) A self-adjoint matrix;
  c) An inner product on a vector space over $\mathbb{C}$.

(4) Which of the following define an inner product?
  a) $\langle A, B\rangle = tr(A + B)$ on $M_{2\times 2}(R)$;
  b) $\langle f, g\rangle = \int_0^{1/2} f(t)\bar{g}(t)dt$ on $C[0, 1]$;
  c) $\langle f, g\rangle = \int_0^1 f(t)\bar{g}(t)w_1(t)dt + \int_0^1 f'(t)\bar{g}'(t)w_2(t)dt$ on $C^\infty[0, 1]$ (the space functions on $[0, 1]$ having derivatives of all orders) where $w_1(t) > 0, w_2(t) > 0$ are continuous positive functions on $[0, 1]$.

(5) Mark true or false. If true, give an argument why, if false, give a counterexample. Let V be a finite dimensional vector space.
  a) If $T: V \to V$ satisfies $|Tv| = |v|$ for any $v \in V$ then T is normal.
  b) The adjoint of a normal operator is normal.
  c) If $T: V \to V$ is a self-adjoint linear operator satisfying $\langle Tx, x\rangle > 0$ for any $x \neq 0$ then $\langle x, y\rangle_2 = \langle Tx, y\rangle$ is an inner product on V.
  d) Every orthonormal set of vectors is linearly independent;
  e) If $T: V \to V$ satisfies $|\lambda| = 1$ for any eigenvalue λ of T then T is unitary.

(6) Let $T: V \to V$ be a normal operator satisfying $T^2 = I$. Show that there exists a subspace $W \subset V$ such that $T(x) = x$ for any $x \in W$ and $T(x) = -x$ for any $x \in W^\perp$.

(7) Let A be a real $n \times n$ matrix all of whose eigenvalues are real. Show that A is normal if and only if it is symmetric.

(8) Let $A = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}$. Write A as $QDQ^\star$ where D is diagonal and Q is unitary.

(9) Let $H \leq G$ be a subgroup of index $[G : H] = 2$.
  Prove that H is a normal subgroup of G.

(10) Let $\sigma \in S_n$ be a permutation. Define an $n\times n$ matrix P_σ by the formula $(P_\sigma)_{ij} = \delta_{i\sigma(j)}$. Prove that $\sigma \mapsto P_\sigma$ is a homomorphism and $\det P_\sigma = \text{sign}(\sigma)$.

(11) Let G be a group. For any $a, b \in G$ *the commutator* of a and b is defined as $[a, b] = aba^{-1}b^{-1}$. Further, denote by $[G, G]$ the subgroup of G generated by the commutators $[a, b]$ where $a, b \in G$ are arbitrary.
  Prove that $[G, G] \trianglelefteq G$ is a normal subgroup of G.

(12) Let $T\colon V \to V$ be a skew-adjoint linear operator. Prove that all the eigenvalues of T are purely imaginary (i.e every eigenvalue λ has the form $\lambda = ia$ where a is real).

(13) Let $f\colon M_{n\times n}(\mathbb{C}) \to \mathbb{C}$ be a function satisfying the following conditions

(a) $f(Id) = 1$

(b) $f(A') = -f(A)$ if A is obtained by interchanging two rows of A.

(c) $f(A') = f(A)$ if A' is obtained from A by adding a multiple of a row to another row.

Is it true that $f(A) = \det A$? If true give a proof, if false, give a counterexample.