(1) Give a proof by induction of the following statement used class:

Let $m > 1$ be a natural number. Then for any $n \geqslant 0$ there exists an integer r such that $0 \leqslant r < m$ and $n \equiv r \pmod m$.

(2) Let p be prime.

Prove that for any natural number a we have $a^{p^{p-1}} \equiv a(\mod p)$.

(3) Let $p = 11$. For every $a = 1, \ldots 10$ find a natural number b such that $1 \leqslant b \leqslant 10$ and $a \cdot b \equiv 1(\mod 11)$.

(4) (a) Find $2^{3^{100}}(\mod 5)$

(b) Find the last digit of $2^{3^{100}}$.

Hint: use part a) but remember that 10 is not prime.

(5) Find $1 + 2 + 2^2 + 2^3 + \ldots + 2^{219}(\mod 13)$.