Solutions to selected problems from homework 10

(1) (a) Prove that $\sqrt[3]{\pi}$ is not constructible.

Solution

Suppose $\sqrt[3]{\pi}$ is constructible. Then $\pi = \sqrt[3]{\pi} \cdot \sqrt[3]{\pi} \cdot \sqrt[3]{\pi}$ is constructible too. However, we know that π is transcendental and hence not constructible. This is a contradiction and therefore, $\sqrt[3]{\pi}$ is not constructible. $\square$

(b) Prove that π^3 is not constructible.

Solution

Suppose π^3 is constructible. Then it must be algebraic because by a theorem from class all constructible numbers are algebraic. This means that there exists a polynomial $P(x) = a_n x^n + \ldots + a_1 x + a_0$ such that $a_n \neq 0$, all a_i's are integers and $P(\pi^3) = 0$. Then $0 = P(\pi^3) = a_n(\pi^3)^n + a_{n-1}(\pi^3)^{n-1} + \ldots + a_1(\pi^3) + a_0 = a_n \pi^{3n} + a_{n-1}(\pi)^{3(n-1)} + \ldots + a_1(\pi^3) + a_0$. In other words, π is a root of the polynomial $Q(x) = a_n x^{3n} + a_{n-1}x^{3n-3} + \ldots + a_1 x^3 + a_0$. This means that π is algebraic which we know is false. This is a contradiction and hence π^3 is not constructible. $\square$

(2) Find a polynomial with integer coefficients which has $1 + \sqrt{2} + \sqrt{3}$ as a root.

Solution

Let $x = 1 + \sqrt{2} + \sqrt{3}$. Then $x - 1 - \sqrt{2} = \sqrt{3}$. taking squares of both sides we get $(x-1)^2 - 2(x-1)\sqrt{2} + 2 = 3$. We can rewrite this as $(x-1)^2 - 1 = 2(x-1)\sqrt{2}$. Again taking squares of both sides we get

$((x-1)^2 - 1)^2 = 4(x-1)^2 \cdot 2$, $(x^2 - 2x)^2 = 8(x^2 - 2x + 1)$, $x^4 - 4x^3 + 4x^2 = 8x^2 - 16x + 8$, $x^4 - 4x^3 - 12x^2 + 16x - 8 = 0$.

Thus x satisfies $x^4 - 4x^3 - 12x^2 + 16x - 8 = 0$.

(3) Find the cardinality of the set of all surds.

Solution

Let S be the set of all surds. It was proved in class that all surds are algebraic. Therefore $|S| \leq |A|$ where A is the set of all algebraic numbers. By another theorem from class A is countable and hence $|S| \leq |A| = |\mathbb{N}|$. This means that S is countable too. On the other hand, it's obvious that $\mathbb{N} \subset S$. Therefore $|N| \leq |S|$. By above $|S| \leq |\mathbb{N}|$. By the Schroeder-Berenstein theorem this implies that $|S| = |\mathbb{N}|$.

(4) Is there a line in the plane such that every point on it is constructible?

Solution

It was proved in class that a point (x, y) is constructible if and only if both x and y are constructible. In other words, the set of constructible points on the plane is equal to $S \times S$ where S is the set of constructible numbers. By a theorem from class the set of constructible numbers is equal to the set of all surds which is countable by the previous problem. Therefore, $|S \times S| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$ is also countable. Thus, the set of constructible points on the plane is countable.

On the other hand, we claim that any line on the plane has the same cardinality as $\mathbb{R}$. Indeed, given a line L with the equation $y = kx + b$ the map $f\colon \mathbb{R} \to L$ given by $f(x) = (x, kx + b)$ is 1-1 and onto.Therefore, $|L| = |\mathbb{R}|$. Vertical lines (they are given by equations $x = c$) can be easily seen to have the same cardinality too.

Thus, since $|\mathbb{R}| > |\mathbb{N}|$, there is no line that consists entirely of constructible points.

(5) Let t be a transcendental number. Prove that the set $F = \{(a + bt) : a, b \in \mathbb{Q}\}$ is not a number field.

Solution

Suppose F is a number field. Since $t \in F$ and F is closed under multiplication we must have that $t^2 = t \cdot t$ is also in F. That means that it can be written as $t^2 = at + b$ for some rational a, b. In other words, t satisfies the quadratic equation $t^2 - at - b = 0$ with rational coefficients. Therefore, t is algebraic. This is a contradiction and hence F is not a number field.