### MAT 246S Solutions to Practice Term Test 1 Winter 2013

(1) Prove by mathematical induction that  $n^3 + 5n$  is divisible by 6 for any natural n.

### Solution

We first check that the statement is true for n=1. We have  $1^3 + 5 = 6$  is divisible by 6.

Suppose the statement is true for  $n \ge 1$ . Let's show that it's also true for n + 1. We have  $(n+1)^3 + 5(n+1) = n^3 + 3n^2 + 3n + 1 + 5n + 5 = (n^3 + 5n) + 3n^2 + 3n + 6$ . Clearly  $3n^2 + 3n + 6 \equiv 0 \pmod{3}$ . Also, either n or n = 1 is even so that n(n+1) is even and hence is divisible by 2. therefore  $3n^2 + 3n + 6 \equiv 3n(n+1) + 6 \equiv 0 \pmod{6}$ . Therefore  $(n+1)^3 + 5(n+1) = (n^3 + 5n) + 3n^2 + 3n + 6 \equiv 0 \pmod{6}$  by induction assumption.

(2) Find the remainder when  $7^{101}$  is divided by 101.

### Solution

Since 101 is prime, By Fermat theorem  $7^{100} \equiv 1 \pmod{101}$  and hence  $7^{107} \equiv 7 \cdot 1 \equiv 7 \pmod{101}$ .

(3) Find the integer  $a, 0 \le a \le 20$  such that  $13a \equiv 1 \pmod{20}$ .

## Solution

We have that  $13 \cdot 3 = 39 \equiv -1 \pmod{20}$ . Hence  $13 \cdot (-3) \equiv 1 \pmod{20}$ . Since  $-3 \equiv 17 \pmod{20}$  we have  $13 \cdot 17 \equiv 1 \pmod{20}$ .

(4) Prove that if  $m \equiv 1 \pmod{\phi(n)}$  and (a, n) = 1 then  $a^m \equiv a \pmod{n}$ , where  $\phi$  is Euler's function.

#### Solution

We are given  $m \equiv 1 \pmod{\phi(n)}$ , i.e  $m = k\phi(n) + 1$  By Euler's theorem  $a^{\phi(n)} \equiv 1 \pmod{n}$ . (mod n). Therefore,  $a^{k\phi(n)} \equiv 1 \pmod{n}$  and hence  $a^{k\phi(n)+1} \equiv 1 \cdot a \equiv a \pmod{n}$ 

(5) Suppose  $3^{3^{100}}$  is written in ordinary way. What are the last two digits?

# Solution

We need to find the remainder when we divide  $3^{3^{100}}$  by 100. Let  $n = 100 = 2^2 \cdot 5^2$ . Then  $\phi(n) = (2^2 - 2^1) \cdot (5^2 - 5^1) = 40$ . therefore, by the previous problem,  $3^{40k+1} \equiv 3 \pmod{100}$ . Next observe that  $3^4 = 81 \equiv 1 \pmod{40}$ . Therefore,  $3^{100} = (3^4)^{25} \equiv 1 \pmod{40}$ . This finally implies that  $3^{3^{100}} \equiv 3 \pmod{100}$ . This means that the last two digits of  $3^{3^{100}}$  are 03. (6) Prove that  $\sqrt[3]{\frac{2}{7}}$  is irrational.

## Solution

Suppose  $\sqrt[3]{\frac{2}{7}} = \frac{a}{b}$  where a, b are integers. we can assume that (a, b) = 1. Then  $\frac{2}{7} = \frac{a^3}{b^3}$  and  $2b^3 = 7a^3$ . LHS is even which means that a must be even. Hence a = 2c and we have  $2b^3 = 7 \cdot 8c^3$ ,  $b^3 = 28c^3$ . Now RHS is even and hence b must be even. That means that both a and b are even which contradicts (a, b) = 1.