(1) Prove that $|\mathbb{Z}| = |\mathbb{N}|$ by constructing an explicit bijection $f\colon \mathbb{N} \to \mathbb{Z}$

(2) Construct an explicit bijection $f\colon [0,1] \to (0,1)$.

(3) For any set S define $P(S)$ to be the set of all subsets of S. For example, if $S = \{a, b\}$ then $P(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

Let A be a finite set. Show that $|P(A)| = 2^{|A|}$.

Hint: Let $A = \{x_1, \ldots, x_n\}$. Represent a subset S of A by a sequence of 0s and 1s of length n such that the i-th element in the sequence is 1 if $x_i \in S$ and is 0 if $x_i \notin S$.

(4) Finish the proof of the Shroeder-Berenstein Theorem from class.

Recall that the proof goes as follows. Let $f\colon S \to T$ and $g\colon T \to S$ be 1-1. Define S_S to be the set of points in S whose last ancestor is in S. Define S_T to be the set of points whose last ancestor is in T. Define S_∞ to be the set of points in S which have infinitely many ancestors. Define T_S, T_T, T_∞ similarly.

Then define $h\colon S \to T$ by the formula

$$h(s) = \begin{cases} f(x) \text{ if } s \in S_S \cup S_\infty \\ g^{-1}(s) \text{ if } s \in S_T \end{cases}$$

It was shown in class that $h\colon S_S \to T_S$ is 1-1 and onto.

Finish the proof by showing that $h\colon S_T \to T_T$ and $h\colon S_\infty \to T_\infty$ are also 1-1 and onto.

(5) Let S be infinite and $A \subset S$ be finite.

Prove that $|S| = |S \backslash A|$.

(6) Let $S = (0,1)$ and $T = [0,1)$. Let $f\colon S \to T$ be given by $f(x) = x$ and $g\colon T \to S$ be given by $g(x) = \frac{x+1}{2}$.

  (a) Find $S_S, S_T, S_\infty, T_S, T_T, T_\infty$

  (b) give an explicit formula for a 1-1 and onto map $h\colon S \to T$ coming from f and g using the proof of the Schroeder-Berenstein theorem.

(7) Let $S = P(\mathbb{N})$

Show that $|S| \leq |\mathbb{R}|$.

Hint: represent a subset A of N as a sequence of 1s and 0s such that the n-th element of the sequence is 1 if $n \in A$ and is 0 if $n \notin A$.