

- (1) Which of the following is a number field?
- (a) the set of all nonnegative rational numbers;
 - (b) the set of numbers of the form $a + b\sqrt{2} + c\sqrt{3}$ where $a, b, c \in \mathbb{Q}$;
 - (c) the set of numbers of the form $a + b\sqrt{2} + c\sqrt[4]{2} + d\sqrt[4]{8}$ where $a, b, c, d \in \mathbb{Q}$.
Hint: Look at the appropriate tower of fields $\mathbb{Q} = F_0 \subset F_1 = \mathbb{Q}(\sqrt{2}) \subset F_2$
 - (d) The set of irrational numbers.
- (2) Let x_0 be a root of the polynomial $a_n x^n + \dots a_1 x + 0$ where each a_i has the form $a_i = b_i + c_i \sqrt{2}$ where $b_i, c_i \in \mathbb{Q}$.
 Prove that x_0 is a root of a polynomial with rational coefficients.
Hint: Write $f(x_0) = 0$, move all the terms with $\sqrt{2}$ to the right and square the sides.
- (3) Let F be the field consisting of real numbers of the form $p + q\sqrt{2 + \sqrt{2}}$ where p, q are of the form $a + b\sqrt{2}$, with a, b rational. Represent

$$\frac{1 + \sqrt{2 + \sqrt{2}}}{2 - 3\sqrt{2 + \sqrt{2}}}$$

in this form.