(1) Prove that

$$\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots+\frac{n}{2^n}=2-\frac{n+2}{2^n}$$

(2) Prove that

$$1+2q+3q^2+\ldots+nq^{n-1}=\frac{1-(n+1)q^n+nq^{n+1}}{(1-q)^2}$$

(3) The Fibonacci sequence is the sequence of numbers $F(1), F(2), \ldots$ defined by the following recurrence relations:
$F(1) = 1, F(2) = 1$, $F(n) = F(n-1) + F(n-2)$ for all $n > 2$.
For example, the first few Fibonacci numbers are $1, 1, 2, 3, 5, 8, 13, \ldots$

Use induction to prove the identity:

$$F(n+1)F(n-1) - F(n)^2 = (-1)^n$$

for all natural numbers $n \geq 2$.

(4) Let $x_1 > 2$. Define x_n by the formula $x_{n+1} = \frac{3x_n+2}{x_n+2}$

Prove that $x_n > 2$ for all n.

(5) Using the method from class find the formula for the sum

$$1^3 + 2^3 + \ldots + n^3$$

Then prove the formula you've found by mathematical induction.

(6) Find a mistake in the following "proof".

Claim. Any two natural numbers are equal.

We'll prove the following statement by induction in n: Any two natural numbers $\leq n$ are equal.

We prove it by induction in n.

a) The statement is trivially true for $n = 1$.

b) Suppose it's true for $n \geq 1$. Let a, b be two natural numbers $\leq n+1$. Then $a - 1 \leq n$ and $b - 1 \leq n$. Therefore, by the induction assumption

$$a - 1 = b - 1$$

Adding 1 to both sides of the above equality we get that $a = b$. Thus the statement is true for $n + 1$. By the principle of mathematical induction this means that it's true for all natural n. $\square$.

(7) Using the method from class write a table of all prime numbers ≤ 100. Explain why you only need to cross out the numbers divisible by $2, 3, 5$ and 7.