MAT 246S

Winter 2012

(1) Find the formula for the sum $1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - \ldots + (2n) \cdot (2n-1) - (2n) \cdot (2n+1)$ and prove it by mathematical induction.

Solution

Observe that $(2n)(2n-1) - (2n)(2n+1) = (2n) \cdot (-2) = -4n$ Thus we need to find $-4 \cdot 1 - \ldots - 4n = -4(1 + \ldots n) = -4\frac{n(n+1)}{2} = -2n(n+1)$. We prove this by induction. When n = 1 we have $1 \cdot 2 - 2 \cdot 3 = 2 - 6 = -4 = -2 \cdot (1) \cdot (2) = -4$. Induction step. Suppose $1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - \ldots - (2n) \cdot (2n+1) = -2n(n+1)$ then $1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - \ldots - (2n) \cdot (2n+1) + (2n+1) \cdot (2n+2) - (2n+2) \cdot (2n+3) = -2n(n+1) + (2n+1) \cdot (2n+2) - (2n+2) \cdot (2n+3) = -2n(n+1) - 2(2n+2) = -2(n+1)(n+2)$.

(2) Find the remainder when 6^{100} is divided by 14.

Solution

First we observe that $6 \equiv -1 \pmod{7}$. Hence $6^{100} \equiv (-1)^{100} = 1 \pmod{7}$. Thus $6^{100} \equiv 1 \pmod{7} \equiv 8 \pmod{7}$. This means that 7 divides $6^{100} - 8$. But $6^{100} - 8$ is even and 2 also divides $6^{100} - 8$. Since (2,7) = 1 this means that 14 divides $6^{100} - 8$, i.e. $6^{100} \equiv 8 \pmod{14}$.

Answer: 8.

(3) Find the integer $a, 0 \le a < 37$ such that $(34!)a \equiv 1 \pmod{37}$.

Solution

Since 37 is prime, by Wilson's theorem, $36! \equiv -1 \pmod{37}$.

We rewrite $34! \cdot 35 \cdot 36 \equiv -1 \pmod{37}$. Since $36 \equiv -1 \pmod{37}$ this gives $34! \cdot 35 \equiv 1 \pmod{37}$.

Answer: a = 35.

(4) Let n = pq where p, q are distinct odd primes. Find the remainder when $\phi(n)!$ is divided by n.

Solution

Since p and q are distinct odd, without loss of generality $2 . We have <math>\phi(n) = (p-1)(q-1)$. Since q > p > 2 we have $\phi(n) = (p-1)(q-1) > (p-1)$ and hence $\phi(n) \ge p$. Similarly, $\phi(n) = (p-1)(q-1) > (q-1)$ and hence $\phi(n) \ge q$. Therefore both p and q occur as factors in the product $\phi(n)! = 1 \cdot 2 \dots \cdot p \cdot \dots \cdot q \cdot \dots \cdot \phi(n)$. Hence n = pq divides $\phi(n)!$ i.e.

Answer: $\phi(n)! \equiv 0 \pmod{n}$.

(5) Find all integer solutions of the equation

$$34x + 50y = 2$$

Solution

First we simplify 17x + 25y = 1. Note that (17, 25) = 1. We compute it using Euclidean algorithm.

 $25 = 1 \cdot 17 + 8, 17 = 2 \cdot 8 + 1$. Hence $8 = 25 \cdot 1 - 17 \cdot 1$ and $1 = 17 \cdot 1 - 2 \cdot 8$. plugging in the former equation into the latter we get $1 = 17 \cdot 1 - 2(25 \cdot 1 - 17 \cdot 1) = 17 \cdot 3 - 25 \cdot 2$. Hence $x_0 = 3, y_0 = -2$ is one solution. Recall if x_0, y_0 solves ax + by = c with (a, b) = 1 then $x = x_0 + kb, y = y_0 - ka$ with $k \in \mathbb{Z}$ is the general integer solution of ax + by = c for any integer k.

In our case this gives

Answer: x = 3 + 25k, y = -2 - 17k with $k \in \mathbb{Z}$ is the general integer solution of 34x + 50y = 2.