MAT 246S **Solutions to Practice Term Test 1** **Winter 2012**

(1) Prove by mathematical induction that $n^3 + 5n$ is divisible by 6 for any natural n.

Solution

We first check that the statement is true for n=1. We have $1^3 + 5 = 6$ is divisible by 6.

Suppose the statement is true for $n \geq 1$. Let's show that it's also true for $n+1$. We have $(n+1)^3+5(n+1) = n^3+3n^2+3n+1+5n+5 = (n^3+5n)+3n^2+3n+6$. Clearly $3n^2+3n+6 \equiv 0 \pmod 3$. Also, either n or $n=1$ is even so that $n(n+1)$ is even and hence is divisible by 2. therefore $3n^2+3n+6 = 3n(n+1)+6 \equiv 0 \pmod 2$. Taken together the above means that $3n^2+3n+6 \equiv 0 \pmod 6$. Therefore $(n+1)^3+5(n+1) = (n^3+5n)+3n^2+3n+6 \equiv 0 \pmod 6$ by induction assumption.

(2) Find the remainder when 7^{101} is divided by 101.

Solution

Since 101 is prime, By Fermat theorem $7^{100} \equiv 1$ (mod 101) and hence $7^{107} \equiv 7{\cdot}1 \equiv 7$ (mod 101).

(3) Find the integer a, $0 \leq a \leq 20$ such that $13a \equiv 1(mod\ \ 20)$.

Solution

We have that $13 \cdot 3 = 39 \equiv -1 \pmod{20}$. Hence $13 \cdot (-3) \equiv 1 \pmod{20}$. Since $-3 \equiv 17 \pmod{20}$ we have $13 \cdot 17 \equiv 1 \pmod{20}$.

(4) Prove that if $m \equiv 1(mod\ \ \phi(n))$ and $(a,n) = 1$ then $a^m \equiv a(mod\ \ n)$, where ϕ is Euler's function.

Solution

We are given $m \equiv 1(mod\ \ \phi(n))$, i.e $m = k\phi(n)+1$ By Euler's theorem $a^{\phi(n)} \equiv 1 \pmod n$. Therefore, $a^{k\phi(n)} \equiv 1 \pmod n$ and hence $a^{k\phi(n)+1} \equiv 1 \cdot a \equiv a \pmod n$

(5) Suppose $3^{3^{100}}$ is written in ordinary way. What are the last two digits?

Solution

We need to find the remainder when we divide $3^{3^{100}}$ by 100. Let $n = 100 = 2^2 \cdot 5^2$. Then $\phi(n) = (2^2-2^1)\cdot(5^2-5^1) = 40$. therefore, by the previous problem, $3^{40k+1} \equiv 3 \pmod{100}$. Next observe that $3^4 = 81 \equiv 1 \pmod{40}$. Therefore, $3^{100} = (3^4)^{25} \equiv 1 \pmod{40}$. This finally implies that $3^{3^{100}} \equiv 3 \pmod{100}$. This means that the last two digits of $3^{3^{100}}$ are 03.

(6) Prove that $\sqrt[3]{\frac{2}{7}}$ is irrational.

Solution

Suppose $\sqrt[3]{\frac{2}{7}} = \frac{a}{b}$ where a, b are integers. we can assume that $(a, b) = 1$. Then $\frac{2}{7} = \frac{a^3}{b^3}$ and $2b^3 = 7a^3$. LHS is even which means that a must be even. Hence $a = 2c$ and we have $2b^3 = 7 \cdot 8c^3$, $b^3 = 28c^3$. Now RHS is even and hence b must be even. That means that both a and b are even which contradicts $(a, b) = 1$.