(2) Let $S = (0,1)$ and $T = [0,1)$. Let $f\colon S \to T$ be given by $f(x) = x$ and $g\colon T \to S$ be given by $g(x) = \frac{x+1}{2}$.

(a) Find $S_S, S_T, S_\infty, T_S, T_T, T_\infty$

(b) give an explicit formula for a 1-1 and onto map $h\colon S \to T$ coming from f and g using the proof of the Schroeder-Berenstein theorem.

Solution

(a) We claim that $S_\infty = T_\infty = \emptyset, S_T = \{1 - \frac{1}{2^n} |\text{ for } n \geq 1\}$, $S_S = (0,1)\backslash S_T$, $T_T = \{1 - \frac{1}{2^n} |\text{ for } n \geq 0\}$ and $T_S = (0,1)\backslash T_T$.
First we observe that $f(1 - \frac{1}{2^n}) = 1 - \frac{1}{2^n}$ for any $n \geq 1$ and $g(1 - \frac{1}{2^n}) = 1 - \frac{1}{2^{n+1}}$ for any $n \geq 0$. So that $g(0) = 1 - \frac{1}{2} = \frac{1}{2}, f(\frac{1}{2}) = \frac{1}{2}, g(\frac{1}{2}) = 1 - \frac{1}{2^2} = \frac{3}{4}, f(\frac{3}{4}) = \frac{3}{4}, g(\frac{3}{4}) = 1 - \frac{1}{2^3} = \frac{7}{8}$ etc. Arguing by induction we see that the last ancestor of $1 - \frac{1}{2^n} \in S$ for any $n \geq 1$ and of $1 - \frac{1}{2^n} \in S$ for any $n \geq 0$ is $y = 0 \in T$ (observe that 0 is not in the image of f and so has no ancestors). Therefore $\{1 - \frac{1}{2^n} |\text{ for } n \geq 1\} \subset S_T$ and $\{1 - \frac{1}{2^n} |\text{ for } n \geq 0\} \subset T_T$.
Next we claim that for any $n \geq 0$ the interval $(1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}})$ is contained in S_S. We will prove this by induction in n.
When $n = 0$ the interval in question is $(0, \frac{1}{2})$. Observe that $g([0,1)) = [\frac{1}{2}, 1)$. Hence if $x \in (0, \frac{1}{2})$ then it has no ancestors and therefore $(0, \frac{1}{2}) \subset S_S$ by definition of S_S.
Induction step. Suppose we already proved that $(1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}})$ is contained in S_S for some $n \geq 0$.
Let $x \in (1 - \frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+2}})$. Then $x = g(y)$ for $y = 2x - 1$ with $y \in (1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}})$ so that $y \in T$ is the first ancestor of x. Then $y = f(y)$ so that $y \in S$ is the second ancestor of x. Clearly, the last ancestor of x is the same as the last ancestor of y. But $y \in S_S$ by the induction assumption and therefore $x \in S_S$ as well.
This completes the induction step and proves that all x not of the form $1 - \frac{1}{2^n}$ belong to S_S.
Therefore, $S_\infty = T_\infty = \emptyset, S_T = \{1 - \frac{1}{2^n} |\text{ for } n \geq 1\}$, $S_S = (0,1)\backslash S_T$.
The same argument works for finding T_T, T_S and T_∞.

(b) By the proof of the Schroeder-Berenstein theorem and part (a) we define $h\colon S \to T$ by the formula

$$h(x) = \begin{cases} f(x) = x \text{ if } x \in S_S \\ g^{-1}(x) = 2x - 1 \text{ if } x \in S_T \end{cases}$$

In other words,

$$h(x) = \begin{cases} x \text{ if } x \neq 1 - \frac{1}{2^n} \\ 1 - \frac{1}{2^{n-1}} \text{ if } x = 1 - \frac{1}{2^n} \end{cases}$$

(3) Let $T = \{1, 2, 3\}$.
Let S be the set of all functions $f\colon \mathbb{N} \to T$.
Prove that $|S| = |\mathbb{R}|$.
Hint: Use problem 7 from homework 8.

Solution

Close a 1-1 map from $\{0, 1\}$ to T. For example, take $h\colon \{0, 1\} \to \{1, 2, 3\}$ given by $h(0) = 1, h(1) = 2$.

This map induces a map ϕ from $\{f\colon N \to \{0, 1\}\}$ to $\{g\colon N \to \{1, 2, 3\}\}$ given by the formula $\phi(f) = h \circ f$. It's easy to see that ϕ is 1-1. Therefore $|\{f\colon N \to \{0, 1\}\}| \le |\{g\colon N \to \{1, 2, 3\}\}|$.

But $\{f\colon N \to \{0, 1\}\}$ can be identified with $P(\mathbb{N})$ which by problem 7 from homework 8 has the same cardinality as $\mathbb{R}$.

Therefore, $|\mathbb{R}| \le |\{f\colon \mathbb{N} \to T\}|$.

On the other hand, we claim that $|\{f\colon \mathbb{N} \to T\}| \le |\mathbb{R}|$.

To see this observe that a function $f\colon N \to T$ can be thought of as a sequence $f(1), f(2), f(3), f(4), \ldots$ where each $f(i)$ is 1,2 or 3.

For every such sequence consider the real number with the decimal expression $0.f(1)f(2)f(3)\ldots$ and define $\psi\colon \{f\colon \mathbb{N} \to T\} \to \mathbb{R}$ given by $\psi(f) = 0.f(1)f(2)f(3)\ldots$. This map is clearly 1-1 and therefore $|\{f\colon \mathbb{N} \to T\}| \le |\mathbb{R}|$.

Finally, by the Schroeder-Berenstein theorem we conclude that $|\{f\colon \mathbb{N} \to T\}| = |\mathbb{R}|$

(4) Let S be an infinite set such that $|S| > |\mathbb{N}|$. Let $T \subset S$ be countable.
  (a) Prove that $S \backslash T$ is infinite.
  (b) Prove that $|S| = |S \backslash T|$.
  Hint: Construct $T' \subset S \backslash T$ such that $|T'| = |\mathbb{N}|$ and use that $|T \cup T'| = |T'|$ to construct an 1-1 and onto map from S to $S \backslash T$.
  (c) Find the cardinality of the set of transcendental numbers.

Solution

(a) Suppose $A = S \backslash T$ is finite. Then $S = A \cup T$. Since $|A| \le |\mathbb{N}|$ and $|T| \le |\mathbb{N}|$ this implies that $|S| \le |\mathbb{N}|$. This is a contradiction as we are given that $|S| > |\mathbb{N}|$.

(b) Since $S \backslash T$ is infinite by part a), we can construct an infinite countable subset $T' \subset S \backslash T$. Let $A = S \backslash (T \cup T')$.
Note that $T \cap T'$ is countable since both T and T' are countable.
Thus, $|T'| = |\mathbb{N}| = |T \cap T'|$. Therefore we can construct a 1-1 and onto map $f\colon T \cup T' \to T'$.
Finally, define $F\colon S = T \cup T' \cup A \to S \backslash T = T' \cup A$ by the formula

$$F(s) = \begin{cases} f(s) \text{ if } x \in T \cup T' \\ s \text{ if } s \in A \end{cases}$$

By construction, F is 1-1 and onto.

(c) Let $S = \mathbb{R}$ and T be the set of all algebraic numbers. Then T is countable and $|S| = |\mathbb{R}| > |\mathbb{N}|$. The set of transcendental numbers is $S \backslash T$. Applying b) we conclude that $|S \backslash T| = |S| = |\mathbb{R}|$.

(5) Let S be the set of sequences $q_1, q_2, q_3, \ldots$ where q_i is real for every i and such that for every sequence there exists $n \in \mathbb{N}$ such that $q_i = 0$ for all $i \geq n$.

Prove that $|S| = |\mathbb{R}|$.

Solution

Let S_n be the set of sequences of the form $q_1, \ldots, q_n, 0, 0, \ldots$. Then $S = \cup_{n=1}^{\infty} S_n$. Let $f_n \colon S_n \to \mathbb{R}^n$ be given by $f(q_1, \ldots, q_n, 0, 0, \ldots) = (q_1, \ldots, q_n)$. Clearly, f_n is a bijection and hence, $|S_n| = |\mathbb{R}^n| = |\mathbb{R}|$ for any n.

Now the result will follow from the following general

Claim. Suppose $T = \cup_{n=1}^{\infty} T_n$ and $|T_n| = |\mathbb{R}|$ for any n.

Then $|T| = |\mathbb{R}|$.

The proof of the claim is the same as the proof of the theorem from class that the union of countably many countable sets is countable.

Clearly $|T| \geq |T_1| = |\mathbb{R}|$. Next we change T_n to $\tilde{T}_n = T_n \backslash \cup_{i=1}^{n-1} T_i$. Then $T = \cup_{n=1}^{\infty} \tilde{T}_n$ and $\tilde{T}_i \cap \tilde{T}_j = \emptyset$ for $i \neq j$.

Let $f_n \colon T_n \to |\mathbb{R}|$ be 1-1. Define $f \colon T \to \mathbb{R} \times \mathbb{R}$ by the formula $f(t) = (f_n(t), n)$ for $t \in \tilde{T}_n$. Since $\tilde{T}_i \cap \tilde{T}_j = \emptyset$ for $i \neq j$ the map f is well defined and by construction it is 1-1. Hence $|T| \leq |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$. By the Schroeder-Berenstein theorem we conclude that $|T| = |\mathbb{R}|$.