(1) Prove that $|\mathbb{Z}| = |\mathbb{N}|$ by constructing an explicit bijection $f\colon \mathbb{N} \to \mathbb{Z}$
(2) Construct an explicit bijection $f\colon [0,1] \to (0,1)$.
(3) Prove that $|\mathbb{N}^n| = |N|$ and $|\mathbb{R}^n| = |\mathbb{R}|$ for any natural n.
(4) For any set S define $P(S)$ to be the set of all subsets of S. For example, if $S = \{a, b\}$ then $P(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

Let A be a finite set. Show that $|P(A)| = 2^{|A|}$.

Hint: Let $A = \{x_1, \ldots, x_n\}$. Represent a subset S of A by a sequence of 0s and 1s of length n such that the i-th element in the sequence is 1 if $x_i \in S$ and is 0 if $x_i \notin S$.

(5) Let S be an infinite set. Prove that $|S| \geq |\mathbb{N}|$.
(6) Let $S = \{(x, y) \in \mathbb{R}^2|$ such that $x^2 + y^2 \leq 1\}$ and $T = \{(x, y) \in \mathbb{R}^2|$ such that $x > 0\}$.

Prove that $|S| = |T|$.

(7) Let $S = P(\mathbb{N})$ - the set of all subsets of $\mathbb{N}$.
   (a) Show that $|S| \leq |\mathbb{R}|$.
   Hint: represent a subset A of N as a sequence of 1s and 0s such that the n-th element of the sequence is 1 if $n \in A$ and is 0 if $n \notin A$.
   (b) Show that $|\mathbb{R}| \leq |S|$.
   Hint: Given a real number $x \in (0,1)$ consider its decimal expression $x = 0.a_1a_2a_3, \ldots$ and form the following set $S(x) = \{a_1, a_1 + 10a_2, a_1 + 10a_2 + 10^2a_3, \ldots\} \subset \mathbb{N}$

Conclude that $|S| = |\mathbb{R}|$.