(1) Prove that for any real numbers $a < b$ there exists an irrational number c such that $a < c < b$.

Hint: Look at the numbers of the form $\frac{p}{q}\sqrt{2}$.

Solution

First observe that we can assume that $0 \notin (a, b)$. If $0 \in (a, b)$ then take $a_1 = 0, b_1 = b$. Since $(a_1, b_1) \subset (a, b)$ it's enough to prove the result for (a_1, b_1).

From now on we assume $0 \notin (a, b)$. Pick an integer n such that $n > \frac{\sqrt{2}}{b-a}$. Then $\frac{\sqrt{2}}{n} < b - a$.

Consider the numbers $0, \pm 1 \cdot \frac{\sqrt{2}}{n}, \pm 2 \cdot \frac{\sqrt{2}}{n}, \ldots, \pm m \cdot \frac{\sqrt{2}}{n} \ldots$.

Pick the largest integer m such that $m \cdot \frac{\sqrt{2}}{n} \leq a$. Then $(m+1) \cdot \frac{\sqrt{2}}{n}$ lies between a and b since otherwise we must have $m \cdot \frac{\sqrt{2}}{n} \leq a < b \leq (m+1) \cdot \frac{\sqrt{2}}{n}$ which implies $\frac{\sqrt{2}}{n} > b - a$ which is false.

Thus there is an integer m such that $a < \frac{m\sqrt{2}}{n} < b$. By our assumption $m \neq 0$ and hence $\frac{m\sqrt{2}}{n}$ is irrational.