(1) Give a careful proof by induction that the Euclidean algorithm always allows to express (a, b) as $(a, b) = ax + by$ for some integer x, y.

   Hint: Use induction in the number of steps in the Euclidean algorithm.

(2) Let a, b, c be natural numbers.

   Let x_0, y_0 be integers satisfying $ax_0 + by_0 = c$.

   Find the general integer solution of the equation $ax + by = c$.

   Note: a and b are not assumed to be relatively prime.

(3) (a) Find $gcd(165, 63)$ using the Euclidean algorithm.

   (b) Write $gcd(165, 63)$ as $gcd(165, 63) = 165x_0 + 63y_0$ for some integer x_0, y_0 *using the Euclidean algorithm.*

   (c) Find the general integer solution of the equation
   $165x + 63y = 18$.

(4) Without using the uniqueness of prime factorization prove that if d is a common divisor of a and b then d divides $gcd(a, b)$.

(5) Let a, b, c be natural numbers. Let (a, b, c) be the largest natural number that divides a, b and c.

   (a) Prove that $(a, b, c) = ((a, b), c)$.

   (b) Prove that the equation $ax+by+cz = (a, b, c)$ has an integer solution.

(6) Let m be a natural number.

   For what values of c does the equation $((m - 1)!)x + my = c$ have an integer solution?

   Hint: the answer depends on whether or not m is prime.