(1) Let p be prime.

Prove that for any natural number a we have $a^{p^{p-1}} \equiv a(\mod p)$.

(2) Let $p = 11$. For every $a = 1, \ldots 10$ find a natural number b such that $1 \leqslant b \leqslant 10$ and $a \cdot b \equiv 1(\mod 11)$.

(3) Find $3^{2^{100}}(\mod 7)$.

(4) (a) Find $2^{3^{100}}(\mod 5)$

(b) Find the last digit of $2^{3^{100}}$.

Hint: use part a) but remember that 10 is not prime.

(5) To what number between 0 and 6 inclusive is the product $11 \cdot 18 \cdot 2322 \cdot 13 \cdot 19$ congruent modulo 7?

(6) Find $1 + 2 + 2^2 + 2^3 + \ldots + 2^{219}(\mod 13)$.

(7) Let $p > 5$ be a prime number. Prove that $6[(p-4)!] \equiv 1(\mod p)$.

(8) Prove that if $gcd(a, b) = 1$ and $a|c$ and $b|c$ then $ab|c$.

Is this true if we don't assume that $gcd(a, b) = 1$?