(1) Prove that

$$\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots+\frac{n}{2^n}=2-\frac{n+2}{2^n}$$

(2) Prove that

$$1+2q+3q^2+\ldots+nq^{n-1}=\frac{1-(n+1)q^n+nq^{n+1}}{(1-q)^2}$$

(3) Find the sum of the following geometric progression

$$\frac{x}{1+x^2}+\frac{x^2}{(1+x^2)^2}+\ldots+\frac{x^n}{(1+x^2)^n}$$

(4) Prove that

$$1^2+3^2+\ldots+(2n+1)^2=\frac{(n+1)(2n+1)(2n+3)}{3}$$

(5) Let $x_1>2$. Define x_n by the formula $x_{n+1}=\frac{3x_n+2}{x_n+2}$
Prove that $x_n>2$ for all n.

(6) Using the method from class find the formula for the sum

$$1^3+2^3+\ldots+n^3$$

Then prove the formula you've found by mathematical induction.

(7) Find a mistake in the following "proof".

Claim. Any two natural numbers are equal.

We'll prove the following statement by induction in n: Any two natural numbers $\leq n$ are equal.

We prove it by induction in n.

a) The statement is trivially true for $n=1$.

b) Suppose it's true for $n\geq 1$. Let a,b be two natural numbers $\leq n+1$. Then $a-1\leq n$ and $b-1\leq n$. Therefore, by the induction assumption

$$a-1=b-1$$

Adding 1 to both sides of the above equality we get that $a=b$. Thus the statement is true for $n+1$. By the principle of mathematical induction this means that it's true for all natural n. $\Box$.

(8) Using the method from class write a table of all prime numbers ≤ 100. Explain why you only need to cross out the numbers divisible by $2,3,5$ and 7.

(9) Let $n\geq 100$ be a composite (i.e. non prime) number. Is it true that there is a prime number $p\leq\sqrt[3]{n}$ that divides n? If true prove it, if false give a counterexample.