

## Practice Final 2

1. Using induction prove that

$$1^2 + 3^2 + \dots + (2n + 1)^2 = \frac{(n + 1)(2n + 1)(2n + 3)}{3}$$

2. Let  $a, b, c$  be natural numbers.

- (a) Show that the equation  $ax + by = c$  has a solution if and only if  $(a, b) | c$ .  
(b) Find all integer solutions of  $6x + 15y = 9$ .

3. Find the last digit of the sum

$$2(1 + 3 + 3^2 + 3^3 + \dots + 3^{309})$$

4. Let  $S$  be infinite and  $A \subset S$  be finite. Prove that  $|S| = |S \setminus A|$ .

5. Let  $S = [0, 1]$  and  $T = [0, 2)$ . Let  $f: S \rightarrow T$  be given by  $f(x) = x$  and  $g: T \rightarrow S$  be given by  $g(x) = x/2$ .

- (a) Find  $S_S, S_T, S_\infty$ ;  
(b) give an explicit formula for a 1-1 and onto map  $h: S \rightarrow T$  coming from  $f$  and  $g$  using the proof of the Schroeder-Berenstein theorem.

6. Let  $n = 2p$  where  $p$  is an odd prime. Find the remainder when  $\phi(n)!$  is divided by  $n$ . Here  $\phi(n)$  is the Euler function of  $n$ .

7. Prove that  $q_1\sqrt{3} + q_2\sqrt{5} \neq q'_1\sqrt{3} + q'_2\sqrt{5}$  for any rational  $q_1, q_2, q'_1, q'_2$  unless  $q_1 = q'_1, q_2 = q'_2$ .

8. Let  $a$  be a root of  $x^5 - 6x^3 + 2x^2 + 5x - 1 = 0$ . Construct a polynomial with integer coefficients which has  $a^2$  as a root.

*Hint:* separate even and odd powers.

9. Find all complex roots of  $x^6 + 7x^3 - 8 = 0$ .

*Reminder:* Real numbers are also complex numbers.

10. Represent  $\sin(5\theta)$  as a polynomial in  $\sin(\theta)$ .

11. Is  $\frac{\sqrt[5]{5} - \sqrt{5}}{1 + 2\sqrt{7}}$  constructible? Justify your answer.

12. For each of the following answer "true" or "false". Justify your answer.

- a) If  $\frac{x}{y}$  is constructible then both  $x$  and  $y$  are constructible.

- b) If  $x$  is constructible then  $\frac{1}{x}$  is constructible.
- c) There is an angle  $\theta$  such that  $\cos \theta$  is constructible but  $\sin \theta$  is not constructible.
- d)  $\sqrt[3]{\frac{10}{27}}$  is constructible.

13. Prove that the equation

$$(1 + x^{19})^3 + (1 + x^{19})^2 - 3 = 0$$

has no constructible solutions.