MAT 246S **Practice Term Test**

(1) Prove by mathematical induction that $n^3 + 5n$ is divisible by 6 for any natural n.

Solution

we check that the statement is true for n=1. we have $1^3 + 5 = 6$ is divisible by 6. Suppose the statement is true for $n \geq 1$. let's show that it's also true for $n + 1$. We have $(n + 1)^3 + 5(n + 1) = n^3 + 3n^2 + 3n + 1 + 5n + 5 = (n^3 + 5n) + 3n^2 + 3n + 6$. Clearly $3n^2 + 3n + 6 \equiv 0 \pmod 3$. Also, either n or $n = 1$ is even so that $n(n + 1)$ is even and hence is divisible by 2. therefore $3n^2 + 3n + 6 = 3n(n + 1) + 6 \equiv 0 \pmod 2$. taken together the above means that $3n^2 + 3n + 6 \equiv 0 \pmod 6$. Therefore $(n + 1)^3 + 5(n + 1) = (n^3 + 5n) + 3n^2 + 3n + 6 \equiv 0 \pmod 6$ by induction assumption.

(2) Find the remainder when 7^{101} is divided by 101.

Solution

Since 101 is prime, By Fermat theorem $7^{100} \equiv 1$ (mod 101) and hence $7^{107} \equiv 7 \cdot 1 \equiv 7$ (mod 101).

(3) Find the integer a, $0 \leq a \leq 20$ such that $13a \equiv 1 (mod\ 20)$.

Solution

We have that $13 \cdot 3 = 39 \equiv -1 \pmod{20}$. Hence $13 \cdot (-3) \equiv 1 \pmod{20}$. Since $-3 \equiv 17 \pmod{20}$ we have $13 \cdot 17 \equiv 1 \pmod{20}$.

(4) Prove that if $m \equiv 1(mod\ \phi(n))$ and $(a, n) = 1$ then $a^m \equiv a(mod\ n)$, where ϕ is Euler's function.

Solution

We are given $m \equiv 1(mod\ \phi(n))$, i.e $m = k\phi(n) + 1$ By Euler's theorem $a^{\phi(n)} \equiv 1 \pmod n$. Therefore,

$a^{k\phi(n)} \equiv 1 \pmod{n}$ and hence $a^{k\phi(n)+1} \equiv 1 \cdot a \equiv a \pmod{n}$

(5) Suppose $3^{3^{100}}$ is written in ordinary way. What are the last two digits?

Hint: Use the previous problem.

Solution

we need to find the remainder when we divide $3^{3^{100}}$ by 100. let $n = 100 = 2^2 \cdot 5^2$. then $\phi(n) = (2^2 - 2^1) \cdot (5^2 - 5^1) = 40$. therefore, by the previous problem, $3^{40k+1} \equiv 3 \pmod{100}$. next observe that $3^4 = 81 \equiv 1 \pmod{40}$. therefore, $3^{100} = (3^4)^{25} \equiv 1 \pmod{40}$. this finally implies that $3^{3^{100}} \equiv 3 \pmod{100}$. This means that the last two digits of $3^{3^{100}}$ are 03.

(6) Prove that $\sqrt[3]{\frac{2}{7}}$ is irrational.

Solution

Suppose $\sqrt[3]{\frac{2}{7}} = \frac{a}{b}$ where a, b are integers. we can assume that $(a, b) = 1$. then $\frac{2}{7} = \frac{a^3}{b^3}$ and $2b^3 = 7a^3$. LHS is even which means that a must be even. hence $a = 2c$ and we have $2b^3 = 7 \cdot 8c^3$, $b^3 = 28c^3$. Now RHS is even and hence b must be even. that means that both a and b are even which contradicts $(a, b) = 1$.

(7) Prove that

$$x = \sum_{n=1}^{\infty} \frac{1}{10^{n^2}}$$

is irrational.

Solution

We know that a number is rational if and only if its decimal expression is periodic. However, the decimal expression for X is not periodic because the gap between n's and $n+1$st ones in its decimal expression is $(n+1)^2 - n^2 = 2n+1$. this number grows arbitrary large with n and hence the decimal expression for x is not periodic.