(1) Prove that the set of functions $f\colon \mathbb{R} \to \mathbb{R}$ has cardinality bigger than $\mathbb{R}$.

**Solution**

for a subset $A \subset \mathbb{R}$ define its characteristic function $\chi_A$ by the formula

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

It's then clear that the map $A \mapsto \chi_A$ gives a 1-1 and onto correspondence between $P(\mathbb{R})$ and functions from $\mathbb{R}$ to $\{0,1\}$. The latter is a subset of functions from $\mathbb{R}$ to $\mathbb{R}$ so it's cardinality is no bigger. Thus we have $|P(\mathbb{R})| = |\{$ functions from $\mathbb{R}$ to $\{0,1\}| \le |\{$ functions from $\mathbb{R}$ to $\mathbb{R}\}|$.

Lastly note that $|\mathbb{R}| < |P(\mathbb{R})|$ by the general theorem from class. Together with the above this yields the result.

(2) Let $S = P(\mathbb{N})$

Show that $|S| = |\mathbb{R}|$.

*Hint:* It was shown on the last homework that $|S| \le |\mathbb{R}|$. By Shroeder-Berenstein it's enough to show that $|\mathbb{R}| \le |S|$. To do this we need to construct a 1-1 map $f\colon \mathbb{R} \to S$. Define $f(x)$ by using the decimal expansion of $x$.

**Solution**

It was proved already that $|S| \le |\mathbb{R}|$ so we only need to prove the opposite inequality. since $|\mathbb{R}| = |[1,\infty)|$ it's enough to construct a $1-1$ map from $[1,\infty)$ to $P(\mathbb{N})$.

Given a real number $x \ge 1$ look at its decimal expansion $n.a_1a_2a_3\ldots$ where $a_i$ are the digits after the decimal. Define $f(x)$ to be the following subset of $\mathbb{N}$: $\{n, na_1, na_1a_2, \ldots\}$. For example if $x = 31.478$ we define $f(x) = \{31, 314, 3147, 31478, 314780, 3147800, \ldots\}$. It's clear that different real numbers will give different subsets. This gives a $1-1$ map from $[1,\infty)$ to $P(\mathbb{N})$.