(1) Let z_0 be a root of $x^n = z$. Show that all roots of $x^n - z = 0$ have the form $z_0 \cdot \zeta_k$ where $\zeta_0, \dots \zeta_{n-1}$ are n-the roots of 1.

Solution

It's easy to see (why?) that if $z = 0$ then the only complex solution of $x^n = z$ is $x = 0$.

Let $z \neq 0$.

If $\zeta^n = 1$ and $z_0^n = z$ then $(z_0 \cdot \zeta)^n = z_0^n \cdot z_0^n = z$. That means that $z_0 \cdot \zeta$ solves $x^n = 1$. Conversely, if $z_1^n = z$ then $(\frac{z_1}{z_0})^n = \frac{z_1^n}{z_0^n} = \frac{z}{z} = 1$. Hence $= \zeta = \frac{z_1}{z_0}$ is a root of 1and $z_1 = \zeta \cdot z_0$.

(2) Prove that $|\mathbb{N}^k| = |\mathbb{N}$ for any natural k.

Hint: Use induction.

Solution

First note that if $|S_1| = |S_2|$ and $|T_1| = |T_2|$ then $|S_1 \times T_1| = |S_2 \times T_2|$.

We will prove the result by induction. The base $k = 1$ is obvious.

Induction step. Suppose the result if proved for $k \geq 1$ and we want to prove it for $k+1$. We have $\mathbb{N}^{k+1} = \mathbb{N}^k \times \mathbb{N}$ and by induction assumption $|\mathbb{N}^k| = |\mathbb{N}|$. By the observation above that implies that $|\mathbb{N}^{k+1}| = |\mathbb{N}^k \times \mathbb{N}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$ where the last equality follows from the theorem proved in class that $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.

(3) For any set S define $P(S)$ to be the set of all subsets of S. for example, if $S = \{a, b\}$ then $P(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

Let A be a finite set. Show that $|P(A)| = 2^{|A|}$.

Hint: Let $A = \{x_1, \dots, x_n\}$. Represent a subset S of A by a sequence of 0s and 1s of length n such that the i-th element in the sequence is 1 if $x_i \in S$ and is 0 if $x_i \notin S$.

Solution 1

Per hint we can identify subsets of A with sequences of 1's and 0's of length n. We just need to count the number of such sequences. We have two choices for the first number, two choices for the second number etc, so the number of such sequences is 2^n.

Solution 2

We prove the result by induction in n. The result is obvious for $n = 1$. Induction step. Suppose we proved it for sets of order $n \geq 1$ and need to prove it for sets of order $n + 1$. Let $S = \{x_1, \dots, x_n, x_{n+1}\}$. All subset of S that don't contain x_{n+1} are exactly the subsets of $\{x_1, \dots, x_n\}$. By the induction assumption there are 2^n such subsets. The subsets of S that do contain x_{n+1} have the form $A \cup \{x_{n+1}\}$ where $A \subset \{x_1, \dots, x_n\}$ Therefore there are also 2^n such subsets. Altogether that gives that $|P(S)| = 2^n + 2^n = 2^{n+1}$.

(4) Let S be an infinite set. Prove that $|S| \geq |\mathbb{N}|$.

Solution

We need to construct a $1-1$ map $f\colon \mathbb{N} \to S$. We will construct it by induction in n. First, since S is infinite it's non-empty so we can pick $s_1 \in S$. Define $f(1) = s_1$. Suppose $f(1), \dots f(n)$ are constructed and are all distinct. The set $S \setminus \{f(1), \dots, f(n)\}$ is non-empty since S is infinite. Pick any $s_{n+1} \in S \setminus \{f(1), \dots, f(n)\}$ and define $f(n+1) = s_{n+1}$. By induction this gives a 1-1 map $f\colon \mathbb{N} \to S$ which means that $|\mathbb{N}| \leq |S|$.