(1) Prove that $\frac{\sqrt{2}+\sqrt[3]{5}}{6}$ is not constructible.

Solution

Suppose $x_0 = \frac{\sqrt{2}+\sqrt[3]{5}}{6}$ is constructible. Since $\sqrt{2}$ and 6 are constructible this implies that $6x_0 - \sqrt{2} = \sqrt[3]{5}$ is constructible.

$\sqrt[3]{5}$ is a solution of the equation $x^3 - 5 = 0$. By a theorem from class if a cubic equation with rational coefficients has a constructible solution it also has a rational one. Therefore $x^3 - 5 = 0$ must have a rational solution. We can write it in the form $\frac{m}{n}$ where m and n are relatively prime. Then by another theorem from class we must have that $m|5$ and $n|1$. Therefore $\frac{m}{n} = \pm 1$ or ± 5. Plugging those numbers into $x^3 - 5 = 0$ we see that none of them are solutions. This is a contradiction and therefore x_0 is not constructible.

(2) Prove that $\frac{\pi^2}{3}$ is not constructible.

Solution

Suppose $x_0 = \frac{\pi^2}{3}$ is constructible. Then $3x_0 = \pi^2$ is also constructible and hence $\pi = \sqrt{3x_0}$ is constructible too. However, every contructible number is algebraic and π is not algebraic. This is a contradiction and hence $\frac{\pi^2}{3}$ is not constructible.

(3) Let F be the field consisting of real numbers of the form $p+q\sqrt{2+\sqrt{2}}$ where p, q are of the form $a + b\sqrt{2}$, with a, b rational.. Represent

$$\frac{1+\sqrt{2+\sqrt{2}}}{2-3\sqrt{2+\sqrt{2}}}$$

in this form.

Solution

$$\frac{1+\sqrt{2+\sqrt{2}}}{2-3\sqrt{2+\sqrt{2}}} = \frac{(1+\sqrt{2+\sqrt{2}})(2+3\sqrt{2+\sqrt{2}})}{(2-3\sqrt{2+\sqrt{2}})(2+3\sqrt{2+\sqrt{2}})}$$

$$= \frac{2+3(2+\sqrt{2})+5\sqrt{2+\sqrt{2}}}{4-9(2+\sqrt{2})} = \frac{8+3\sqrt{2}+5\sqrt{2+\sqrt{2}}}{-14-9\sqrt{2}}$$

$$= \frac{(8+3\sqrt{2}+5\sqrt{2+\sqrt{2}})(-14+9\sqrt{2})}{(-14-9\sqrt{2})(-14+9\sqrt{2})} =$$

$$= \frac{-112-52\sqrt{2}-70\sqrt{2+\sqrt{2}}+72\sqrt{2}+27\cdot 2+45\sqrt{2}\sqrt{2+\sqrt{2}}}{196-2\cdot 81}$$

$$= \frac{-58+20\sqrt{2}+(-70+45\sqrt{2})\sqrt{2+\sqrt{2}}}{4}$$

(4) Find a tower of fields $Q = F_0 \subset F_1 \subset F_2 \subset F_3$ such that

$$\sqrt{1+\sqrt{2}+\sqrt{3}} \in F_3$$

Show that all the steps in the tower except for the last one are nontrivial. I.e show that $F_0 \neq F_1$, and $F_1 \neq F_2$.

Solution

Let $F_0 = \mathbb{Q}, F_1 = F_0(\sqrt{2}), F_2 = F_1(\sqrt{3}), F_3 = F_2(\sqrt{1+\sqrt{2}+\sqrt{3}})$.

Note that $F_1 \neq F_0$ since $\sqrt{2}$ is irrational. To see that $F_2 \neq F_1$ suppose $F_2 = F_1$. Then $\sqrt{3} \in F_1$ and we can write $\sqrt{3}$ as

$$\sqrt{3} = a + b\sqrt{2}$$

where $a, b \in \mathbb{Q}$. It's easy to see that we must have $a \neq 0, b \neq 0$. Squaring both sides of the above formula, we get

$3 = (a+b\sqrt{2})^2 = a^2 + 2b^2 + 2ab\sqrt{2}$. This implies that $\sqrt{2} = \frac{3-a^2-2b^2}{2ab}$ is rational. This is a contradiction and therefore $F_2 \neq F_1$.

(5) Show that none of the real roots of $3x^3 - 2x^2 - 2 = 0$ are constructible.

Solution

This is a cubic equation with rational coefficients.

By a theorem from class if this equation has a constructible root it also has a rational one. Let $x_0 = \frac{m}{n}$ be a rational root where $(m,n) = 1$. Then $m|2$ and $n|3$. Therefore $\frac{m}{n} = \pm 1, \pm 2, \pm\frac{2}{3}, \pm\frac{1}{3}$. Plugging in those numbers into the equation we see that none of them are roots.

(6) Let $0 < \theta < \pi/2$ be the angle with $\cos\theta = \frac{2}{7}$. Show that the angle θ is constructible but $\theta/3$ is not.

Solution

$\cos\theta = 2/7$ is rational which means that the angle θ is constructible. Let $x = \cos(\theta/3)$. Suppose the angle $\theta/3$ is constructible. Then x_0 is a constructible number. By the formula $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ we have that x_0 is a root of $4x^3 - 3x = \frac{2}{7}, 28x^3 - 14x - 2 = 0$.

By the theorem mentioned above if this equation has a constructible root it must also have a rational root. Let $x_1 = \frac{m}{n}$ be a rational root where $(m,n) = 1$. Then $m|2$ and $n|28 = 2 \cdot 2 \cdot 7$. Therefore $\frac{m}{n} = \pm 1, \pm 2, \pm\frac{2}{7}, \pm\frac{1}{7}, \pm\frac{1}{14}, \pm\frac{1}{28}$. Plugging in those numbers into the equation we see that none of them are roots.

(7) Show that the equation

$x^9 - 4x^3 + 1 = 0$ has no constructible roots.

Solution

Suppose x_0 is a constructible root. Then $y_0 = x_0^3$ is also constructible. It satisfies $y^3 - 4y + 1 = 0$. As before, this is a cubic equation with rational coefficients and if it has a constructible root then it also has a rational one. Let $y_1 = \frac{m}{n}$ be a rational root where $(m,n) = 1$. Then $m|1$ and $n|1$. Therefore $\frac{m}{n} = \pm 1$. Plugging in $y = \pm 1$ into $y^3 - 4y + 1 = 0$ we see that none of them are roots.