(1) (a) Let V be a finite dimensional vector space over $\mathbb{C}$ and $A\colon V \to V$ be a linear map. Let $V_{\mathbb{R}}$ be the same space V viewed as a vector space over $\mathbb{R}$ and $A_{\mathbb{R}} = A\colon V_{\mathbb{R}} \to V_{\mathbb{R}}$.
Prove that $tr(A_{\mathbb{R}}) = 2Re(tr(A))$

(b) Let G be a compact Lie group. For any real representation of G of a real vector space U define the real character by

$$\chi_U^{\mathbb{R}}(g) = Tr(l_g))$$

Suppose $U_1 \in Irr(G, \mathbb{R})_{\mathbb{R}}$ and $U_2 \in Irr(G, \mathbb{R})_{\mathbb{H}}$.
Prove that

$$\int_G \chi_{U_1}^{\mathbb{R}}(g)\chi_{U_2}^{\mathbb{R}}(g)dg = 0$$

(2) Define a two dimensional complex representation of $\mathfrak{so}(3)$ as follows

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

(a) Verify that this actually defines a Lie algebra representation.

(b) Prove that this Lie algebra representation does not come from a Lie group representation of $SO(3)$ on $\mathbb{C}^2$.
Hint: If a Lie algebra representation $\Pi\colon \mathfrak{g} \to End(V)$ of $\mathfrak{g} = T_eG$ comes from a Lie group representation of G on V and $exp(X) = e$ for some $X \in \mathfrak{g}$ then $e^{\Pi(X)}$ must be equal to Id.