Boundaries at Infinity

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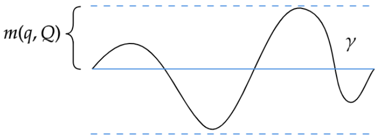
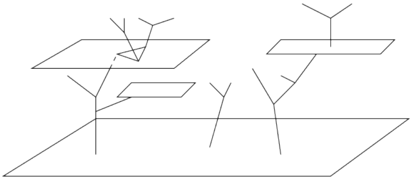
The Sublinearly Morse Boundary

Morse: invariant under quasi-isometry



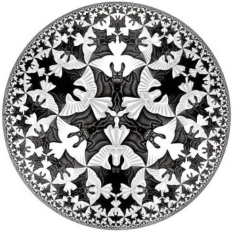
Sublinear: random walks converge to the sublinearly Morse boundary

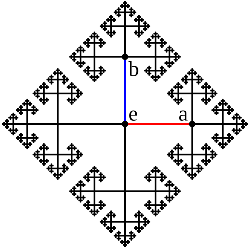


The Gromov Boundary

Definition

The Gromov boundary of a hyperbolic metric space X is the set $\partial X = \{[\gamma] \mid \gamma \text{ is a geodesic ray }\}$.





The topology is generated by the following open neighbourhoods around $[\gamma]$:

$$U([\gamma], r) = \left\{ [\gamma'] \mid \liminf_{s,t \to \infty} (\gamma(s), \gamma'(t))_o \geq r \right\}.$$

Quasi-isometry

Definition

Let (X_1, d_1) and (X_2, d_2) be metric spaces. A function $f : X_1 \to X_2$ is a quasi-isometry if there exists constants $q \geq 1$ and $Q \geq 0$ such that for any two points $x, y \in X_1$,

$$\frac{1}{q} d_1(x, y) - Q \leq d_2(f(x), f(y)) \leq q d_1(x, y) + Q.$$

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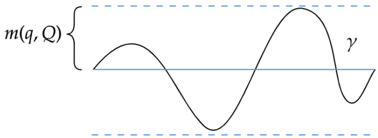
Theorem

Let X_1, X_2 be hyperbolic metric spaces, and let $f : X_1 \to X_2$ be a quasi-isometry. Then f induces a homeomorphism on the Gromov boundaries ∂X_1 and ∂X_2.

Quasi-isometry

Theorem (Morse lemma)

Let X be a hyperbolic space, and γ a (q, Q)–quasi-geodesic in X. Then there is a constant $m(q, Q)$ such that γ is in the $m(q, Q)$–neighbourhood of the geodesic segment connecting its endpoints.



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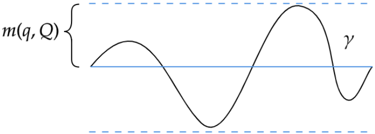
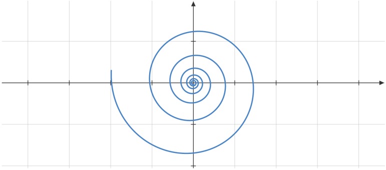
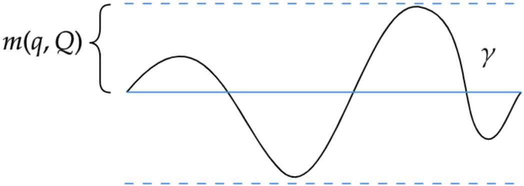




The Morse Boundary

Definition

A geodesic γ is M–Morse if any quasi–geodesics with endpoints on γ is contained in the M–nbhd γ.



Definition (Cashen-Mackay, Charney-Sultan, Cordes)

The Morse boundary of a geodesic metric space X is the set $\partial X = \{[\gamma] \mid \gamma \text{ is a } M\text{–Morse (quasi–)geodesic ray for some } M\}$.

Random Walks

Theorem (Kaimanovich)

In a hyperbolic group G, almost all sample paths $\{x_n\}$ of the random walk (G, μ) converge to a (random) point in the Gromov boundary.

The Tree of Flats

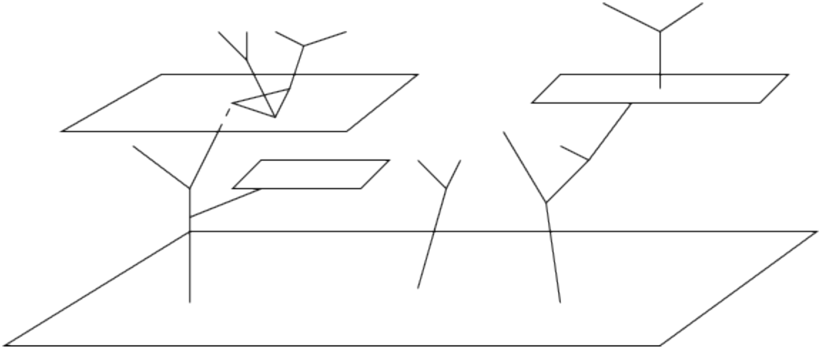
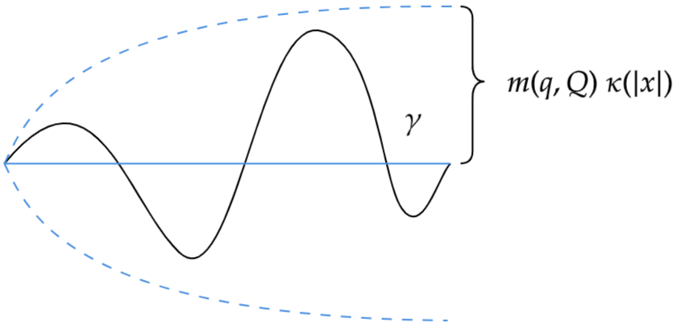


Image by Alex Sisto

The Sublinearly Morse Boundary

Definition (Qing-Rafi-Tiozzo)

A quasi-geodesic γ is sublinearly Morse if every quasi-geodesic β with endpoints on γ is contained in the $M\kappa(|x|)$–nbhd of γ, where M is a constant and κ is a sublinear function.



The Sublinearly Morse Boundary

Definition

The sublinearly Morse boundary of a geodesic metric space X is the set $\partial X = \{[\gamma] \mid \gamma \text{ is a sublinearly Morse quasi–geodesic ray}\}$.

Theorem (H.)

The sublinearly Morse boundary contains the Morse boundary as a topological subspace.

Summary: Boundaries

	Gromov Boundary	Morse Boundary	Sublinearly Morse Boundary
Compact	✓	✗	✗
Metrizable	✓	✓	✓
Invariant Under Quasi-Isometries	✓	✓	✓
Random Walk Converges	✓	✗	✓