Random Walks on Groups and Superlinear Divergent Geodesics

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Random Walks on Groups

Cayley Graph

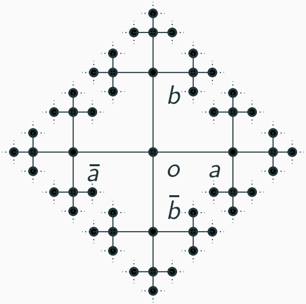

Figure 1: The Cayley Graph of $F_2 = \langle a, b \rangle$.

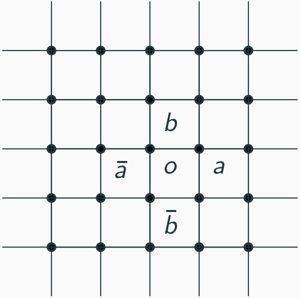

Figure 2: The Cayley Graph of $\mathbb{Z}^2 = \langle a, b \mid [a, b] \rangle$.

Random Walks on Groups

Let G be a finitely generated group acting on a metric space X (usually its Cayley graph). Let μ be a probablity measure on G. Fix $o \in X$.

A random walk is the series of random variables $\{Z_n\}$ defined by

$$Z_n = g_1 g_2 \dots g_n \cdot o,$$

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Example. $G = F_2 = \langle a, b \rangle$, $\mu(g) = \begin{cases} 1/4 & \text{if } g \in \{a, b, \bar{a}, \bar{b}\} \\ 0 & \text{otherwise.} \end{cases}$.

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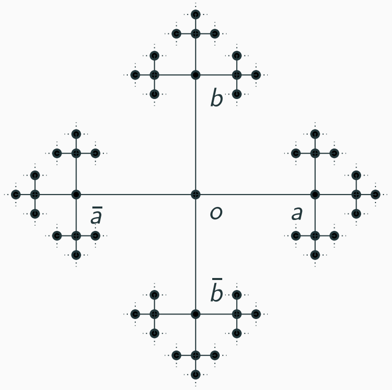

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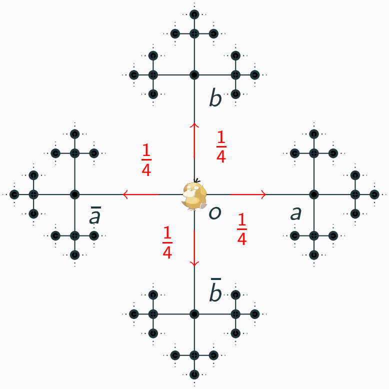

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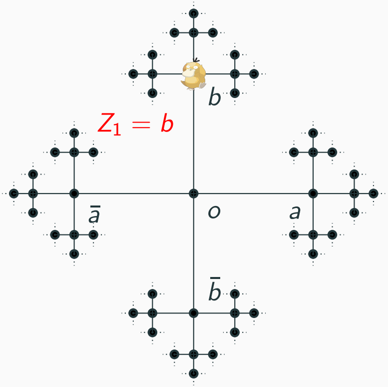

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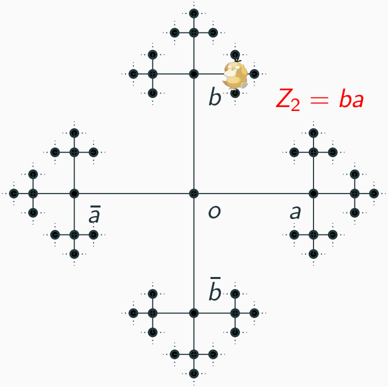

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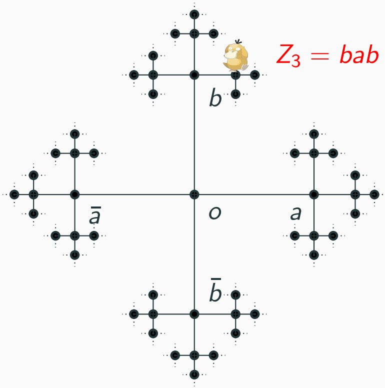

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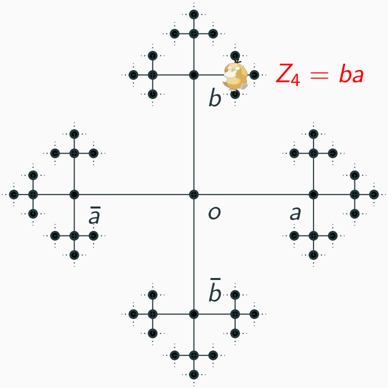

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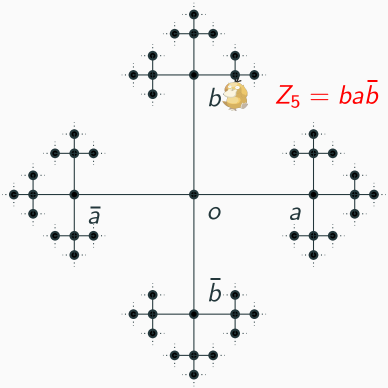
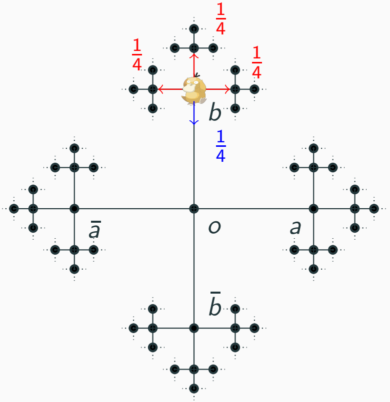


Central Limit Theorem

Question: How quickly is the random walk drifting away from o?



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Theorem (Central Limit Theorem)
Let G be a finitely generated group [under some conditions]. Let $(Z_n)_{n\geq 1}$ be a simple random walk on G. Then there exists constants λ, σ such that

$$\frac{d_X(o, Z_n o) - \lambda n}{\sigma\sqrt{n}} \to \mathcal{N}(0,1) \qquad \textit{in distribution.}$$

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- Free groups (Sawyer-Steger 1987, Ledrappier 2001)

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- Groups with hyperbolic characteristics?

Hyperbolicity and More

Hyperbolicity

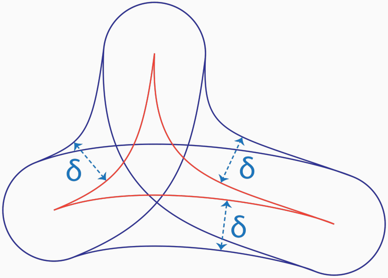


Figure 3: δ–hyperbolicity.

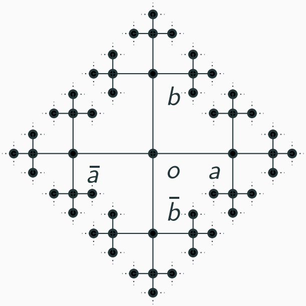


Figure 4: F_2 is 0–hyperbolic.

Hyperbolicity

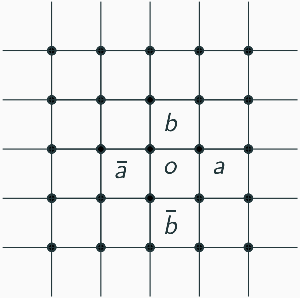


Figure 5: $\mathbb{Z}^2$ is not hyperbolic.

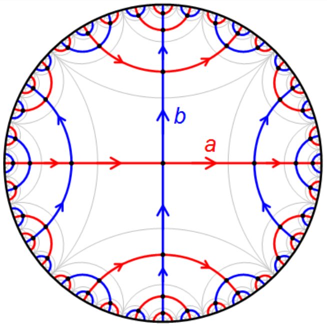
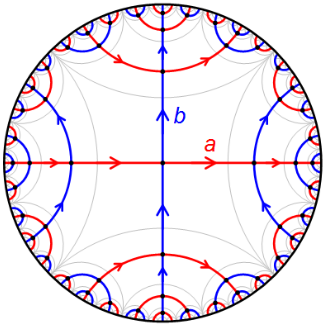


Figure 6: Fuchsian groups are hyperbolic.

Key Geometric Property

Schottky set: powers of loxodromic elements



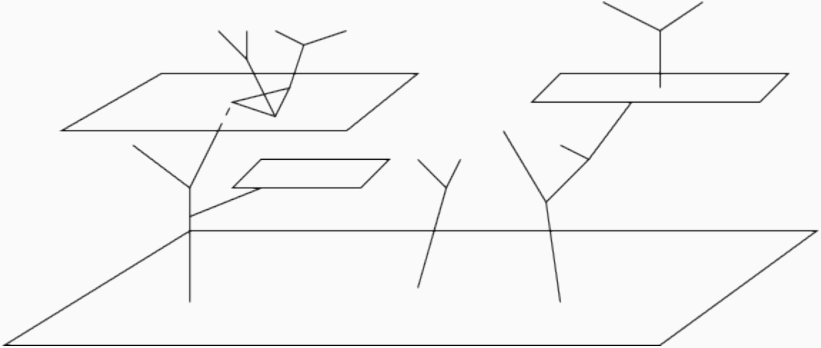


Figure 7: The Cayley graph of $\mathbb{Z}^2 \star \mathbb{Z}$.

Generalizations of Hyperbolicity

Theorem (Mathieu-Sisto, 2019)
Let G be an acylindrically hyperbolic group and μ be a probability measure on G with finite exponential moment. Let $(Z_n)_{n\geq 1}$ be a simple random walk on G. Then there exists constants λ, σ such that

$$\frac{d_X(o, Z_n o) - \lambda n}{\sigma\sqrt{n}} \to \mathcal{N}(0,1) \qquad \textit{in distribution.}$$

Example: Mapping Class Group (acts acylindrically on the curve complex)

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Example: Mapping Class Group (acts acylindrically on the curve complex)

Issue: not invariant under quasi-isometry

Definition (Quasi-isometry)
Let X, Y be metric spaces, a function $f : X \to Y$ is a (q, Q)–quasi-isometry if

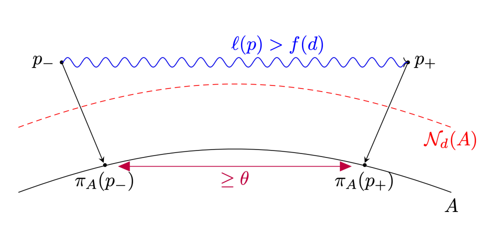
$$\frac{1}{q} d_X(a, b) - Q \leq d_Y(f(a), f(b)) \leq q d_X(a, b) + Q.$$

Superlinear Divergence

Superlinear Divergence

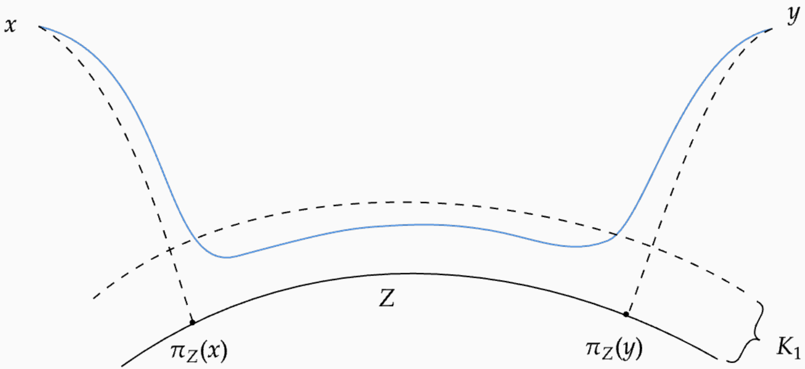
Definition (Goldsborough-Sisto)
A subset A is(f, θ)–divergent if for any $d > 0$ and any path p outside of a d–neighbourhood of A,

$$d(\pi_Z(p_-), \pi_Z(p_+)) > \theta \implies l(p) > f(d).$$



Note: the projection is not the closest point projection, but a coarsely Lipschitz projection

Bottleneck Property



Bottleneck Property: Half or Double

Lemma
Let Z be (f, θ)–divergent. There exists K_0 such that if a geodesic α satisfies

$$d(\alpha(0), Z) > K_0 \qquad \text{and} \qquad d\big(\pi_Z\alpha(0), \pi_Z\alpha(t)\big) \geq \theta.$$

Then either

$$d(\alpha(t), Z) \geq 2 \cdot d(\alpha(0), Z) \qquad \text{or} \qquad d(\alpha(t), Z) \leq \frac{1}{2} \cdot d(\alpha(0), Z).$$

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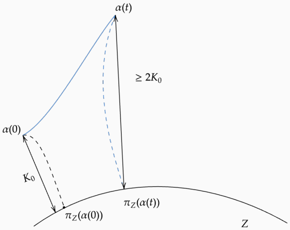
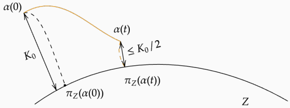




Proof.
Suppose $2K_0 > d(\alpha(t), Z) > K_0/2$, then by superlinear divergence of Z,

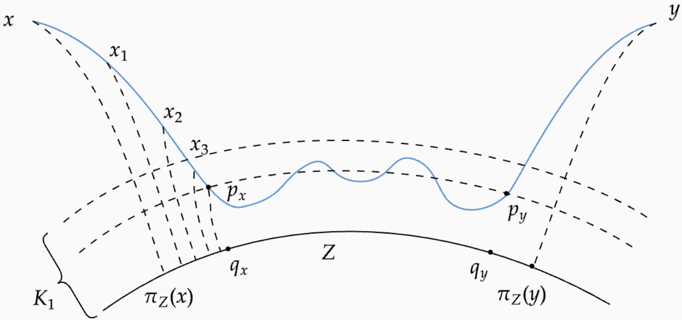
$$d(\alpha(0), \alpha(t)) > f(K_0/2).$$

On the other hand,

$$d(\alpha(0), \alpha(t)) < K_0 + \theta + 2K_0.$$

This fails for sufficiently large K_0. □

Bottleneck Property



Central Limit Theorem

Theorem (Chawla-Choi-H.-Rafi)
Let G be a finitely generated group with exponential growth, and suppose that G has a superlinear-divergent quasi-geodesic. Let $(Z_n)_{n\geq 1}$ be a simple random walk on G. Then there exist constants λ, σ such that

$$\frac{d_X(o, Z_n o) - \lambda n}{\sigma\sqrt{n}} \to \mathcal{N}(0,1) \qquad \text{in distribution.}$$

Application

Groups that contain superlinear divergent geodesics:

- relatively hyperbolic groups
- acylindrically hyperbolic 3-manifold groups
- RACG of thickness at least 2