Square-tiled surfaces and curve counting

Ontario Topology Seminar

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Curve Counting

Curve Counting

X: a hyperbolic surface of genus g and n boundary components.

Question: How many simple closed geodesic curves are there on X of length at most L?

Asymptotic Answer

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Question: How many simple closed geodesics curves are there on X of length at most L? (denoted $N(X, L)$)

Asymptotic answer:

Theorem (Mirzakhani '08)

$$\lim_{L\to\infty} \frac{N(X, L)}{L^{6g-6+2n}} \textit{ exists and is positive.}$$

Follow up: What happens for a specific L?

For specific L (sufficiently large)

We find a lower bound.

Theorem (H. (in progress))
There exists C, L_0 *such that when* $L > L_0$,

$$N(X, L) \geq CL^{6g-6+2n}$$

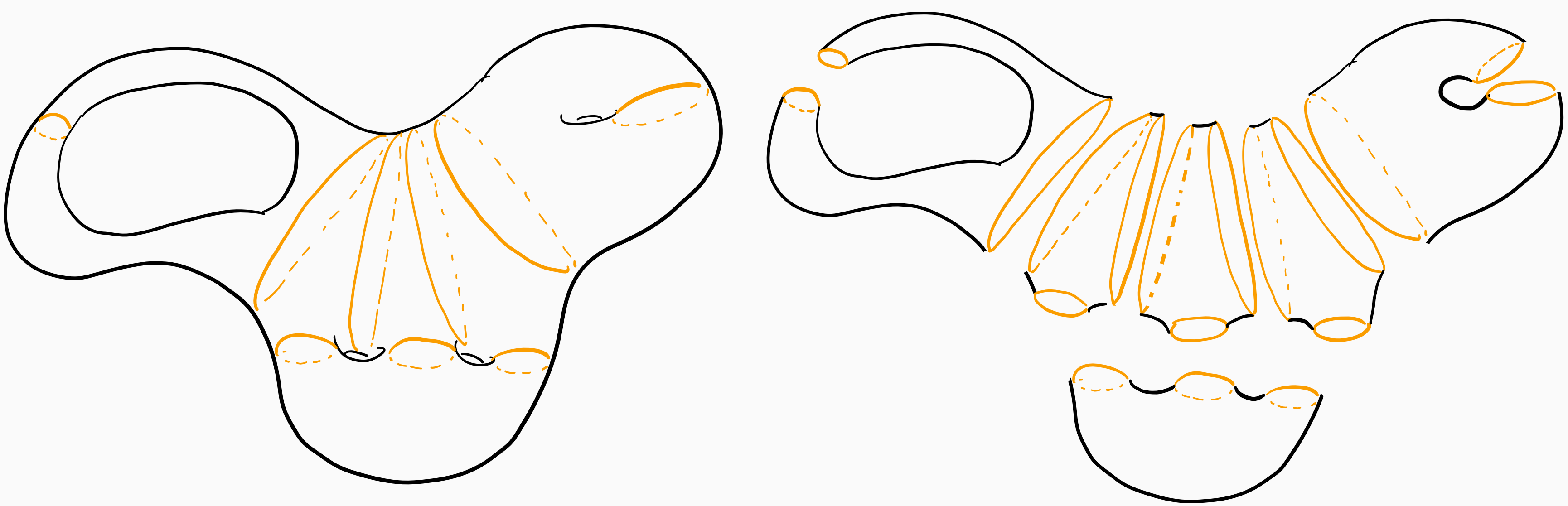
C is independent of the metric

L_0 depends on the systole.

Strategy of Proof

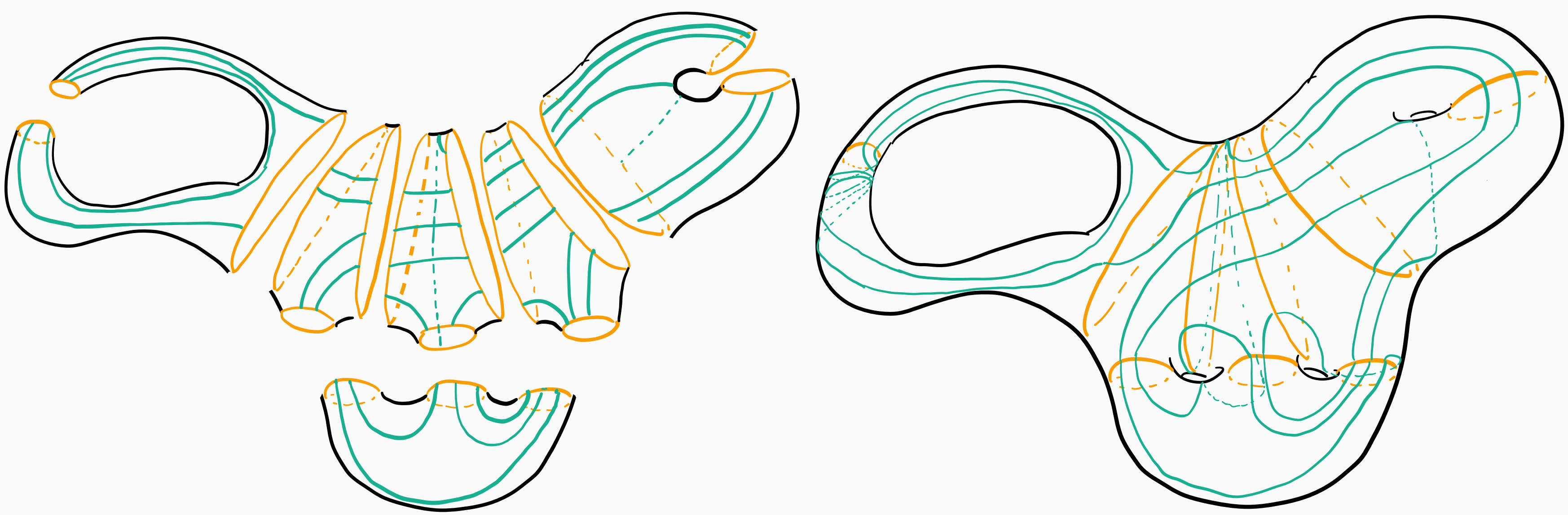
We will construct a collection of simple closed curves as follows:

- Fix a "nice" pair of pants decomposition
- On each pair of pants, construct a collection of arcs



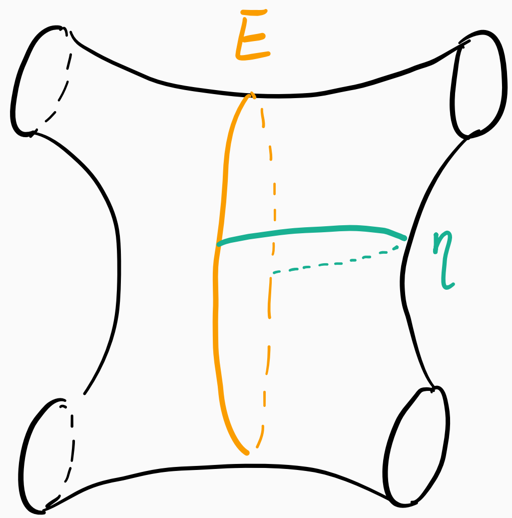
Strategy of Proof

- Glue the arcs together along boundary components via a fractional twist
- This results in a collection of multi-curves. We claim that a positive proportion of them are, in fact, simple.

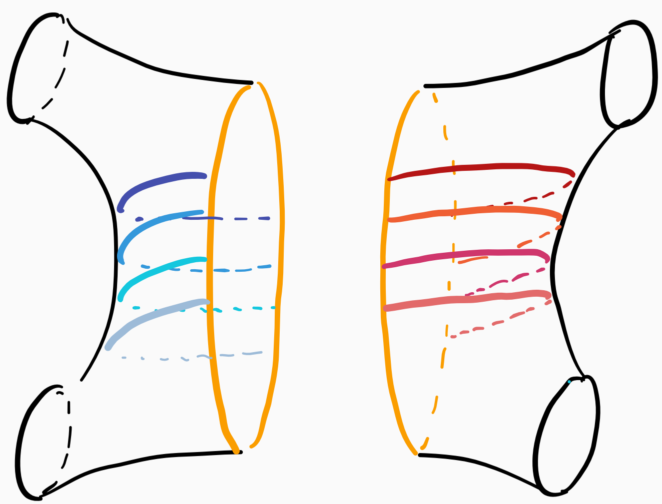


Proof Idea (Special Case: a sphere of 4 boundary components)

(Based upon work of Rivin)

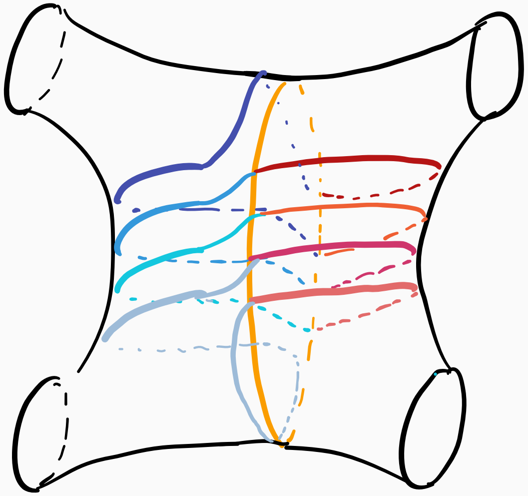


Take k arcs in each pair of pants

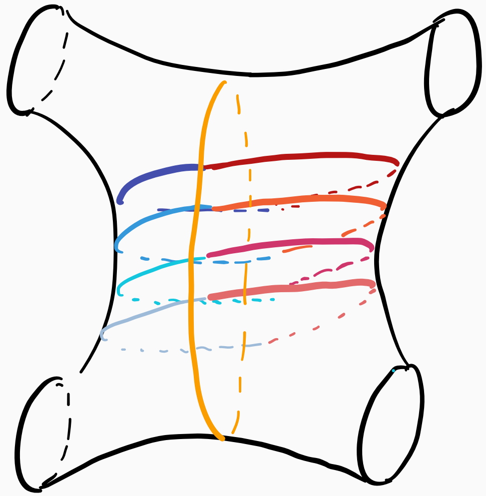


Proof Idea (Special Case: a sphere of 4 boundary components)

Glue them together via a $p/2k$ fractional twist



$p=1 \quad k=4$



$p=0 \quad k=4$

The resulting curve is sometime simple ($p = 1, k = 4$) and sometimes not ($p = 0, k = 4$)

The resulting multicurve is simple if and only if $\gcd(k, p) = 1$

Proof Idea (Special Case: a sphere of 4 boundary components)

Let's count them.

A multi-curve obtained by k copies of η on each side and a $p/2k$ fractional twist along E has length

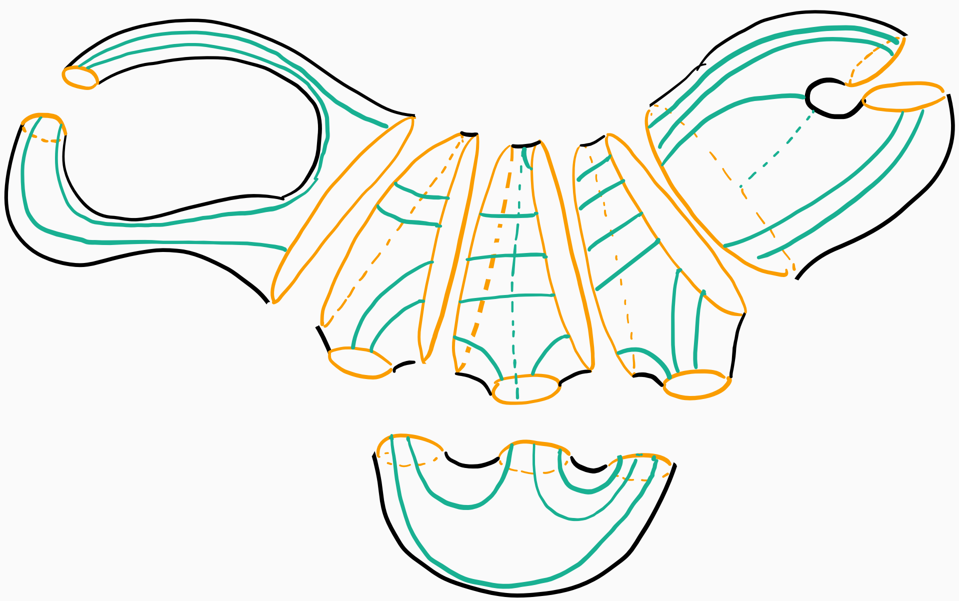
$$2kl(\eta) + pl(E)$$

The multi-curve is simple if and only if $\gcd(k, p) = 1$. So we count

$$\begin{aligned} &\left\{(k,p) \in \mathbb{Z}^2 : 2kl(\eta) + pl(E) \leq L \quad \text{and} \quad \gcd(k,p) = 1\right\} \\ \approx &\frac{6}{\pi^2} \frac{L^2}{4l(\eta)l(E)} \\ \geq &\frac{3}{2\pi^2} L^2. \end{aligned}$$

In general, this fails.

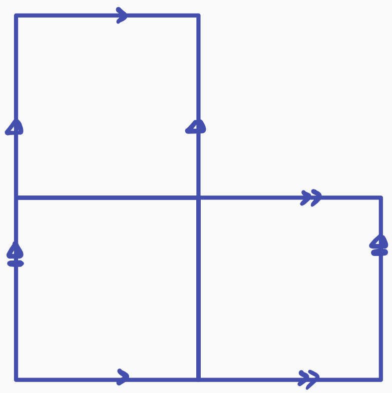
The coprime trick doesn't generalize in higher genus.



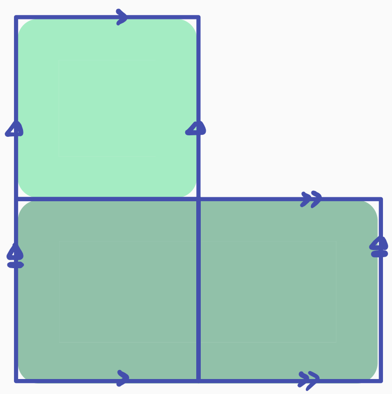
Square-Tiled Surfaces

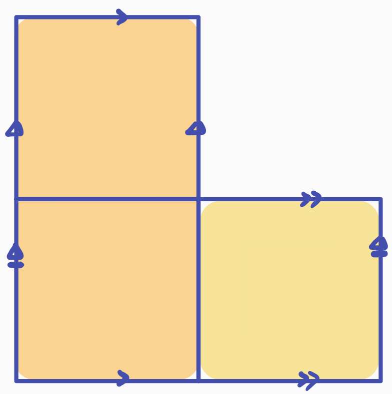
Square-Tiled Surface

A square-tiled surface is a special type of translation surface obtained by gluing together unit squares along their edges by translations. A



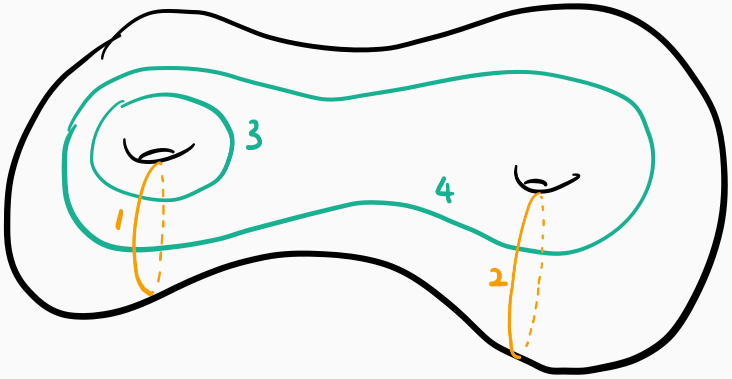
square-tiled surface can be decomposed into horizontal(or vertical) cylinders.

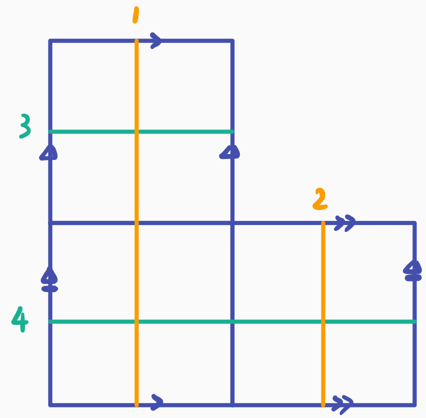




Connection to Curve Counting

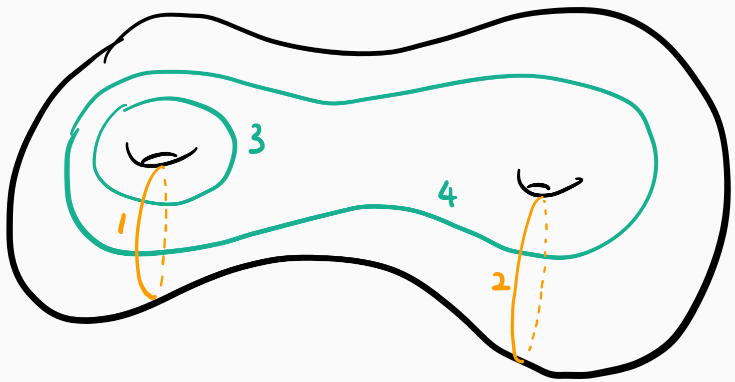
Given a pair of filling multi-curves on a surface, we can construct a square-tiled surface by taking the dual graph.

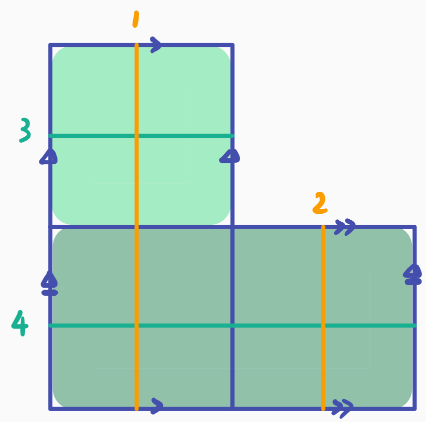




This is a one-to-one correspondence.

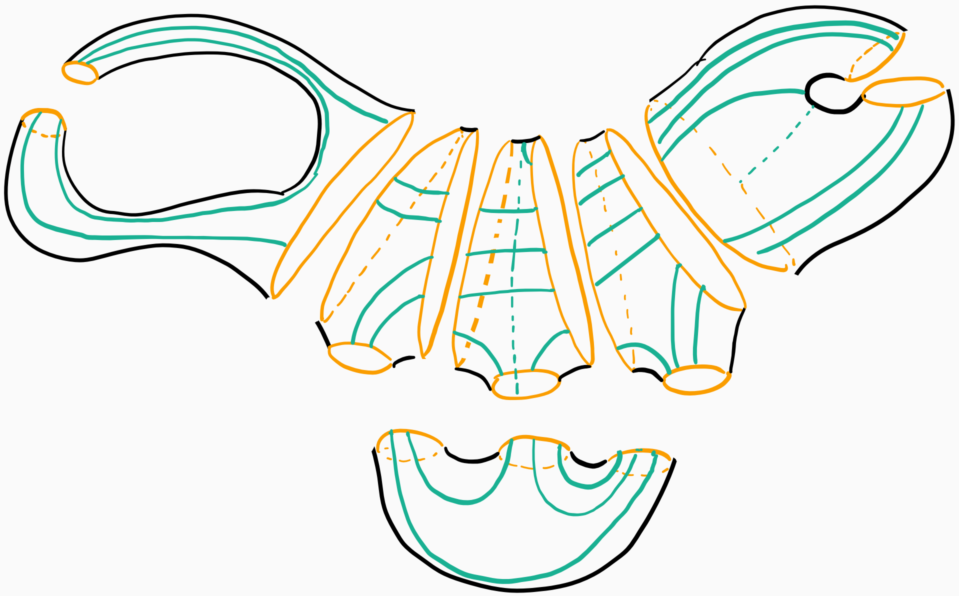
Connection to Curve Counting





The number of components in the horizontal (vertical) multi-curve corresponds to the height (width) of the square-tiled surface.

Connection to Curve Counting



Our question: fix a pair of pants decomposition and arcs on each pair of pants, glue them together, what proportion of the resulting multi-curves are simple?

Translation: fix a horizontal cylinder decomposition, what proportion of square-tiled surfaces with this horizontal cylinder decomposition has height 1?

Connection to Curve Counting

Fix a horizontal cylinder decomposition Γ, what proportion of square-tiled surfaces with this horizontal cylinder decomposition has height 1?

We claim that:

$$\lim_{n\to\infty} \frac{N_1(\Gamma, n)}{N(\Gamma, n)} = \lim_{n\to\infty} \frac{N_1(n)}{N(n)} > 0$$

where

$N(\Gamma, n)$: the number of square-tiled surface with horizontal cylinder decomposition Γ and n squares

$N_1(\Gamma, n)$: the number of square-tiled surface with horizontal cylinder decomposition Γ, n squares, and height 1

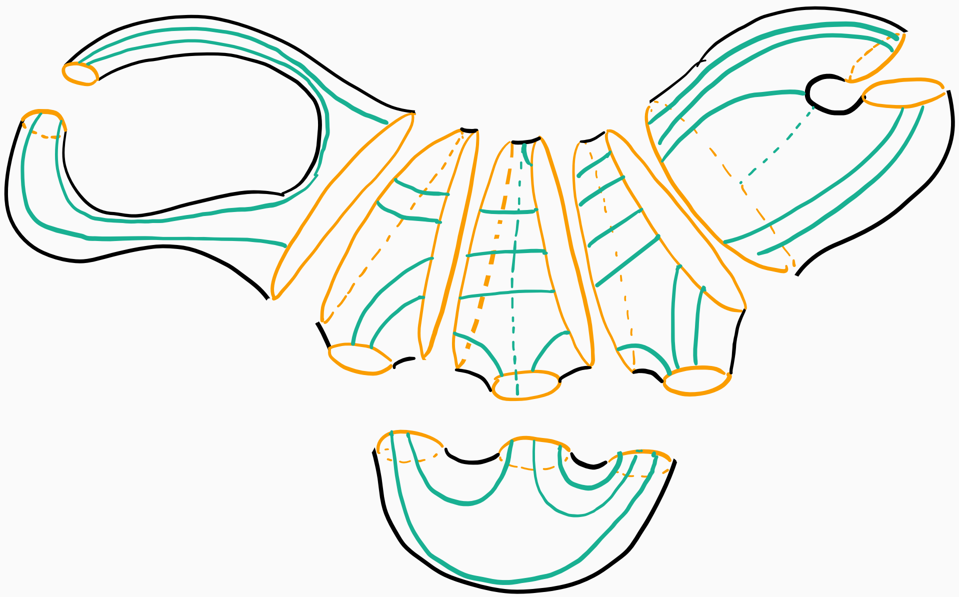
$N(n)$: the number of square-tiled surface with n squares

$N_1(n)$: the number of square-tiled surface with n squares and height 1

This is due to the independence of horizontal and vertical cylinder decomposition, generalizing a theorem of Delecroix-Goujard-Zograf-Zorich.

Heuristics

- The number of copies of arcs contributes to $3g-3+n$ dimensions.
- The number of viable fractional twists contributes to $3g-3+n$ dimensions.



$3g-3+n$ curves in the pants decomposition

Dependence on metric

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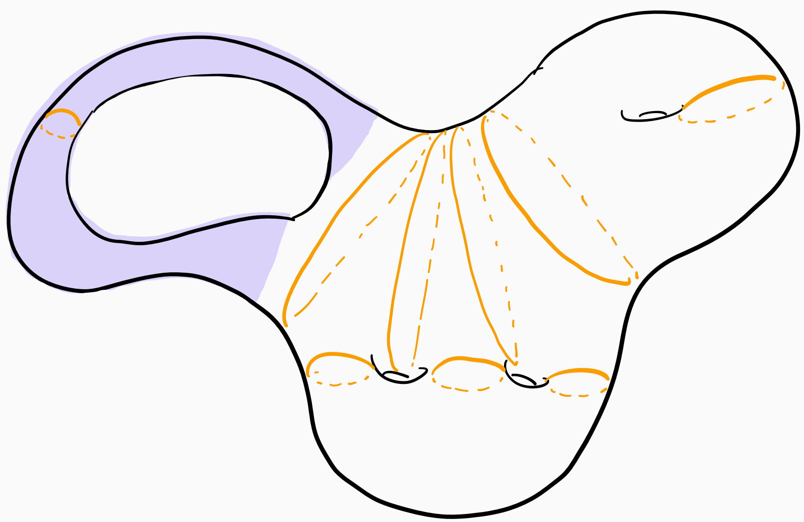

Thank you!

Are there questions?