de Rham Theorem

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Stokes formula and the integration morphism:

Let $M = \bigcup_{\sigma\in\Sigma} \sigma$ be a smooth triangulated manifold.

Fact: Stokes formula $\int_{\partial\sigma} \omega = \int_{\sigma} d\omega$ holds, e.g. for simplices.

It can be used to define linear map Int_k. The map

$Int_{k-1} : \Omega^k(M) \to \Sigma_k^*$ defines a homomorphism of complexes.

Note: Stokes thm. implies commutativity of the diagram:

$$
\begin{array}{ccccccccc}
\dots \longrightarrow & \Omega^{k-1}(M) & \xrightarrow{d_{k-1}} & \Omega^{k}(M) & \xrightarrow{d_{k}} & \Omega^{k+1}(M) & \longrightarrow \dots \\
& Int_{k-1}\Big\downarrow \uparrow \Phi^{k-1} & & Int_{k}\Big\downarrow \uparrow \Phi^{k} & & Int_{k+1}\Big\downarrow \uparrow \Phi^{k+1} & \\
\dots \longrightarrow & \Sigma_{k-1}^* & \xrightarrow{\partial_{k-1}^*} & \Sigma_k^* & \xrightarrow{\partial_k^*} & \Sigma_{k+1}^* & \longrightarrow \dots
\end{array}
$$

Elementary Forms:

If $p_1, p_2, \dots p_s$ are the vertices of complex K , the set $\{St(p_k)\}_k$,

where $St(p_k) := \bigcup_{\sigma:\overline{\sigma}\ni p_k} \sigma$, forms an open cover for M .

The **partition of unity** theorem guarantees the existence of a

C^∞-partition of unity $\phi_1, \dots, \phi_s$ subordinate to $\{St(p_k)\}_k$.

Below we denote the dual basis to simpleces by the same letters.

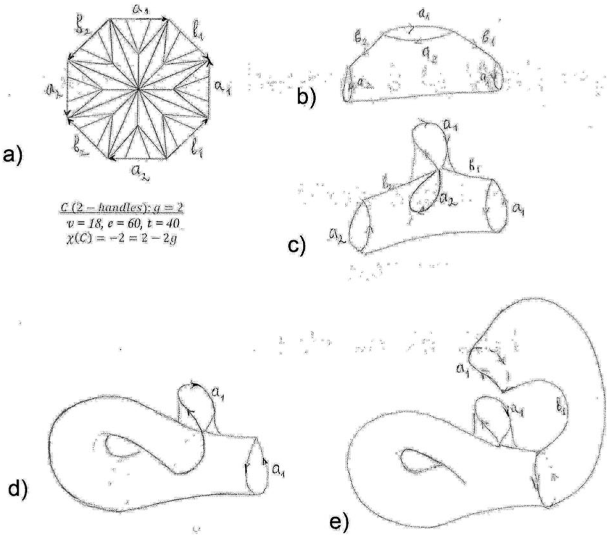
Let $\sigma = [p_{\lambda_0} \dots p_{\lambda_k}]$ be an oriented simplex. Corresponding to

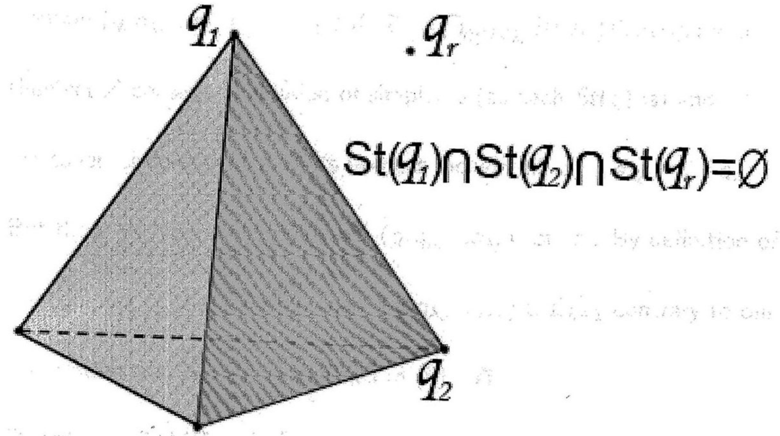
σ is the **elementary differential form** of order k

$\Phi^k(p_{\lambda_0}, \dots, p_{\lambda_k}) = k! \sum_{i=0}^{k} (-1)^i \phi_{\lambda_i} d\phi_{\lambda_0} \wedge \dots \wedge \widehat{d\phi_{\lambda_i}} \wedge \dots \wedge d\phi_{\lambda_k}$.

Triangulation of 2-handles:

v. $= 18$, **e.** $= 60$, **t.** $= 40$ $\Rightarrow$ $\chi(M) = -2$.





q1
.qr
St(q1)∩St(q2)∩St(qr)=∅
q2

1. Right inverse to integration via elementary forms

Proof: induction on k . Below $\omega \cdot \sigma$ is the pairing by integration.

$k = 0 \implies Int^0(\Phi^0(q_i)) \cdot q_j := (\Phi^0(q_i))(q_j) = \Phi_i(q_j) = \delta_{ij}$.

Inductive step $k > 0$: if $\sigma \neq \tau$ then $\tau \subset M \setminus St(\sigma)$ (Lemma 1).

Then $Int^k(\phi^k[\sigma]) \cdot \tau = 0$.

Let $\partial\sigma =: \alpha +$ [other (k-1)-faces of] σ .

Then $\partial^*(\alpha) = \sum_{\partial\tau\supset\alpha} \tau$. So, $\int_\sigma \Phi^k(\sigma) = \int_\sigma \Phi^k \partial^*(\alpha) =$

$\int_\sigma d\Phi^{k-1}(\alpha) = \int_{\partial\sigma} \Phi^{k-1}(\alpha) = \int_\alpha \Phi^{k-1}(\alpha) = 1$.

(Using Stokes, Lemma 2, and the inductive hypothesis)

Two lemmas needed for Step 1.

Remark: $Supp\,\Phi(\sigma) \subset \bigcap_{a \in V(\sigma)} St(a) := St(\sigma)$.

Lemma 1. $\tau \neq \sigma; \tau, \sigma \in \Sigma_k \implies \tau \in M \setminus St(\sigma)$.

Proof. Let $b \in V(\tau) \setminus V(\sigma)$. Either $(b, \sigma) \in \Sigma_{k+1} \implies$ $supp\,\Phi(\sigma) \bigcap St(b) = \emptyset$ or $(b, \sigma) \notin \Sigma_{k+1}$. In the latter case, if $\beta \in \Sigma_{k+1}$, $b \in V(\beta) \implies \sigma \notin \mathcal{F}(\beta) \implies St(b) \bigcap \sigma = \emptyset$. In either case, $\int_\sigma \Phi(\tau) = \int_\tau \Phi(\sigma) = 0$, or, $[\sigma] \cdot \tau = [\tau] \cdot \sigma = 0$.

Lemma 2. $\Phi^k \partial^* = d\,\Phi^{k-1}$. Observe that $\sum \phi_i = 1$, so $\sum d\phi_i = 0$. Also $d\Phi^k(q_{\lambda_0} \dots q_{\lambda_k}) = (k+1)! d\lambda_0 \wedge \dots \wedge d\lambda_k$.

Let $\sigma = q_{\lambda_0} \dots q_{\lambda_k}$, then

$\frac{1}{(k+1)!}\Phi^{k+1}\partial^*[\sigma] = \frac{1}{(k+1)!}\sum_{[q_r\sigma]\in\Sigma_{k+1}} \Phi^{k+1}[q_r\sigma] =$

$\sum_{[q_r\sigma]\in\Sigma_{k+1}}[\phi_{q_r} d\phi_{\lambda_0} \wedge \dots \wedge d\phi_{\lambda_k} + \sum_{i=0}^{k}(-1)^{i+1}\phi_{\lambda_i} d\phi_{q_r} \wedge d\phi_{\lambda_0} \wedge$

$\dots \wedge \widehat{d\phi_{\lambda_i}} \wedge \dots d\phi_{\lambda_k}] =$

$\sum_{[q_r\sigma]\in\Sigma_{k+1}} \phi_{q_r} d\phi_{\lambda_0} \wedge \dots d\phi_{\lambda_k} + \sum_{[q_r\sigma]\notin\Sigma_{k+1}, q_r\notin V(\sigma)} d\phi_{q_r} \wedge$

$\sum_{i=0}^{k}(-1)^{i}\phi_{\lambda_i} d\phi_{\lambda_0} \wedge \dots \wedge \widehat{d\phi_{\lambda_i}} \wedge \dots d\phi_{\lambda_k}] + \sum_{j=0}^{k} d\phi_{\lambda_j} \wedge$

$\sum_{i=0}^{k}(-1)^{i}\phi_{\lambda_i} d\phi_{\lambda_0} \wedge \dots \wedge \widehat{d\phi_{\lambda_i}} \dots \wedge d\phi_{\lambda_k} =$

$d\phi_{\lambda_0} \wedge \dots \wedge \dots d\phi_{\lambda_k} = \frac{1}{(k+1)!} d\Phi^k[\sigma].$

de Rham Complex

$$
\begin{array}{ccccccccc}
\dots & \longrightarrow & ker\ Int_{k-1}(M) & \longrightarrow & ker\ Int^{k}(M) & \longrightarrow & ker\ Int^{k+1}(M) & \longrightarrow & \dots \\
 & & \big\downarrow i & & \big\downarrow i & & \big\downarrow i & & \\
\dots & \longrightarrow & \Omega^{k-1}(M) & \xrightarrow{d_{k-1}} & \Omega^{k}(M) & \xrightarrow{d_k} & \Omega^{k+1}(M) & \longrightarrow & \dots \\
 & & Int_{k-1} \big\downarrow \uparrow \Phi^{k-1} & & Int_{k} \big\downarrow \uparrow \Phi^{k} & & Int_{k+1} \big\downarrow \uparrow \Phi^{k+1} & & \\
\dots & \longrightarrow & \Sigma^*_{k-1} & \xrightarrow{\partial^*_{k-1}} & \Sigma^*_{k} & \xrightarrow{\partial^*_{k}} & \Sigma^*_{k+1} & \longrightarrow & \dots
\end{array}
$$

Let k^{th} de Rham cohomology group $H^k(M) := \ker\ d_k / im\, d_{k-1}$.

Let k^{th} cohomology group of Σ $H^k(\Sigma) := \ker \partial^*_k / im\, \partial^*_{k-1}$.

Note: $Int_k : \Omega^k(M) \to \Sigma^*_k$ induces an isomorphism

$Int_k : H^k(M) \to H^k(\Sigma)$ of differential complexes.

Acyclicity of the kernel of *Int*. map.

Basic fact: **Poincare Lemma**: If U is a contractible open set in

$\mathbb{R}^n$ and α a k-smooth closed form on U, then α is exact,

i.e there exists a form β such that $\alpha = d\beta$.

Observe that $St(\sigma)$ is contractible so Poincare lemma applies.

Next fact that we will need is the extension of forms theorem.

Extension of forms theorem.

(a_k) Let $U(\partial\sigma))$ be ngbhd of of $\partial\sigma$, σ a $s-$simplex,

$\omega \in \Omega^k(U(\partial\sigma))$ closed, $k \geq 0, s \geq 1$.

If $\int_{\partial\sigma} \omega = 0$ and $s = k+1$, then $\exists\, \tilde{\omega} \in \Omega^k(U(\sigma))$ cl.s.t.

$\tilde{\omega}|_{U(\partial\sigma)} = \omega$, perhaps by shrinking $U(\partial\sigma)$.

(b_k) If $s \geq 1$, $k \geq 1, \sigma$ an $s-$simplex, $\omega \in \Omega^k(U(\sigma))$ closed and

$\alpha \in \Omega^{k-1}(U(\partial\sigma))$, $U(\partial\sigma) \subset U(\sigma)$, s.t. $d\alpha = \omega|_{U(\partial\sigma)}$.

When $s = k$ assume $\int_\sigma \omega = \int_{\partial\sigma} \alpha$. Then exists $\tilde{\alpha} \in \Omega^{k-1}(U(\sigma))$

s.t. $\tilde{\alpha}|_{U(\partial\sigma)} = \alpha$ and $d\tilde{\alpha} = \omega$, maybe shrinking $U(\sigma) \supset U(\partial\sigma)$.

Proof of acyclicity, by induction on $s \leq n$.

Consider an s-dim. subcomplex $L_s := \bigcup_i \sigma_i^s$ and $\omega \in ker(Int_k)$ a closed form.

Outline: Construct inductively nbhds $U(L_s)$ of L_s and forms $\alpha_s \in \Omega^{k-1}(U(L_s))$ s.t. $\alpha_s|_{U(L_s)\bigcap U(L_{s-1})} = \alpha_{s-1}$, $d\alpha_s = \omega|_{U(L_s)}$ and $Int^{k-1}(\alpha_{k-1}) = 0$. Then $\alpha_n \in ker(Int_{k-1})$ and $d\alpha_n = \omega$, proving that $ker(Int_\bullet)$ is acyclic.

Proof of acyclicity, by induction.

Basis step:

Choose disjoint, contractible nbds $U(\sigma_i^0)$. By Poincare Lemma exists $\alpha_0' \in \Omega^0(U(\sigma_i^0))$ with $d\alpha_0' = \omega|_{U(\sigma_i^0)}$. Set $\alpha_0 := \alpha_0'$ for $k > 1$ and $\alpha_0 := \alpha_0' - \alpha_0'(\sigma_i^0)$ for $k = 1$ so $Int_0(\alpha_0) = 0$ as required for $s = 0$.

Inductive Step:

Given α_{s-1}, for each σ_i^s we now construct nbds $U(\sigma_i^s)$ s.t. overlaps of each two are subsets of $U(L_{s-1})$ and also forms

Proof of acyclicity, by induction.

$\alpha_{s,i} \in \Omega^{k-1}(U(\sigma_i^s))$ that coincide with α_{s-1} on overlaps. Inductive assumption includes $d\alpha_{s-1} = \omega|_{U(L_{s-1})}$ and $\alpha_{s-1} \in ker(Int_{k-1}(U(L_{s-1})))$ for $s = k$.

Then (b_k) gives $\tilde{\alpha}_{s,i} \in \Omega^{k-1}(U(\sigma_i^s))$ s.t. $d\tilde{\alpha}_{s,i} = \omega|_{U(\sigma_i^s)}$ and $\tilde{\alpha}_{s,i}|_{U(\partial\sigma_i^s)} = \alpha_{s-1}$. Glue $\tilde{\alpha}_{s,i}$ into $\tilde{\alpha}_s$ on $U(L_s) := \cup_i U(\sigma_i^s)$. We set $\alpha_s := \tilde{\alpha}_s$ for $s \neq k-1$ and $\alpha_s := \tilde{\alpha}_s - \Phi^{k-1}(Int_{k-1}(\tilde{\alpha}_s))$ for $s = k-1$.

Proof of acyclicity, by induction (concluded).

Note that $\Phi^{\bullet}$ and $Int_{\bullet}$ are homomorphisms of complexes and the former is the right inverse of the latter by [1] imply

$d\alpha_{k-1} = \omega - \phi^k(Int_k(\omega)) = \omega$ on $U(L_s)$ and also that

$Int_{k-1}(\alpha_{k-1}) = Int_{k-1}(\tilde{\alpha}_{k-1}) - Int_{k-1}(\tilde{\alpha}_{k-1}) = 0$

concluding the proof.

Euler characteristic $\chi(T(M))$ does not depend on the triangulation $T(M)$ of M

Reason: Corollary to the Theorem implies $ker(d'_k) = im(d'_{k-1})$

where d' is the restriction of the exterior derivative in the kernel.

which in turn implies $\frac{ker(\partial_k^*)}{im(\partial_{k-1}^*)} \cong \frac{im(d_k)}{im(d_{k-1})}$.

Note: $\#\{\sigma \in T(M) : dim\sigma = k\} = dim_{\mathbb{R}}\Sigma_k = \dim_{\mathbb{R}} \Sigma^k$.

Theorem: Euler characteristic

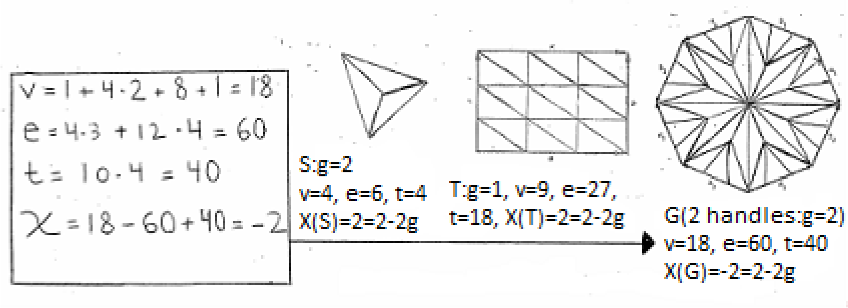
$\chi(M) := \sum_{k=1}^{n}(-1)^k dim_{\mathbb{R}} \frac{ker(d_k)}{im(d_{k-1})} = \chi(T(M))$.

Corollary: $\chi(T(M))$ does not depend on triang. $T(M)$ of M.

Proof of the corollary

$$dim_{\mathbb{R}}\Sigma^k = dim_{\mathbb{R}}(Im(\partial_k^*)) + dim_{\mathbb{R}}\frac{ker(\partial_k^*)}{im(\partial_{k-1}^*)} + dim_{\mathbb{R}}Im(\partial_{k-1}^*).$$

Therefore $\chi(M) = \sum_{k=0}^{n}(-1)^k dim_{\mathbb{R}}\frac{ker(\partial_k^*)}{im(\partial_{k-1}^*)} = \chi(T(M)).$



Extension of Forms Theorem

Proof: by induction on k. **Outline:** Show (a_0) holds, then

$(a_{k-1}) \implies (b_k)$, and finally, $(b_k) \implies (a_k)$.

(a_0) : Say $\omega \in \Omega^0(U(\partial\sigma))$ closed. Then ω is locally constant.

If $s > 1$, then $\omega \equiv \mathit{const}$ in $U(\partial\sigma)$ so we can let let $\tilde{\omega} = \omega$ in $U(\sigma)$.

If $s = 1$ then $\sigma = p_0 p_1$ as an $1-$simplex; also it is given that

$\int_{\partial\sigma} \omega = 0$. But $\int_{\partial\sigma} \omega = \omega(p_1) - \omega(p_0) = 0$ so we can let $\tilde{\omega} = \omega$.

$(a_{k-1}) \implies (b_k)$: Say ω, α are as in (b_k). Poincare lemma gives

$\alpha' \in \Omega^{k-1}(U(\sigma))$, $d\alpha' = \omega|_{U(\sigma)}$. Let $\alpha - \alpha' =: \beta \in \Omega^{k-1}(U(\partial\sigma))$.

Then β is closed in $U(\partial(\sigma))$. If $s = k$ then $\int_{\partial\sigma} \beta = \int_{\partial} \alpha - \int_{\partial\sigma} \alpha'$

$= \int_\sigma \omega - \int_\sigma d\alpha' = 0$. Applying (a_{k-1}) to β we get a closed form

$\tilde{\beta} \in \Omega^{k-1}(U(\sigma))$ such that $\tilde{\beta}|_{U(\partial\sigma)} = \beta$.

Then $\tilde{\alpha} := (\tilde{\beta} + \alpha') \in \Omega^{k-1}(U(\sigma))$ is as required in (b_k).

$(b_k) \implies (a_k)$: Say $\sigma = (p_0 \ldots p_s)$ and ω are as in $(a_k), k > 0$.

Also, let $\sigma' := (p_1 \ldots p_s) \in \mathcal{P}$, where $\mathcal{P}$ is he union of proper faces of σ with p_0 as a vertex. Then ω is defined and closed in a ngbhd $U(\mathcal{P})$; clearly $U(\mathcal{P}) \subset St(p_0)$ and it is star-shaped.

Poincare lemma gives $\alpha' \in \Omega^{k-1}(U(\mathcal{P}))$ s.t. $d\alpha' = \omega|_{U(\mathcal{P})}$; this holds in particular in some nbhd $U(\partial\sigma') \subset U(\mathcal{P})$. For $s = k+1$ define $A := (\partial\sigma - \sigma') \in \Sigma_k$. Then $\partial A = -\partial\sigma'$, and hence $\int_{\sigma'} \omega - \int_{\partial\sigma'} \alpha' = \int_{\sigma'} \omega + \int_A d\alpha' = \int_{\partial\sigma} \omega = 0$. Applying now (b_k) to simplex σ' we get $\tilde{\alpha'} \in \Omega^{k-1}(U(\sigma'))$ such that $\tilde{\alpha'}|_{U(\partial\sigma'))} = \alpha'$ and $d\tilde{\alpha'} = \omega|_{U(\sigma')}$. Shrink $U(\mathcal{P})$ so that $U(\mathcal{P}) \bigcap U(\sigma') \subset U(\partial\sigma')$, let $U(\partial\sigma) := U(\mathcal{P}) \bigcup U(\sigma')$ and set $\tilde{\alpha} \in \Omega^{k-1}(U(\partial\sigma))$ by $\tilde{\alpha} = \alpha'$ on $U(\mathcal{P})$ and $\tilde{\alpha} = \tilde{\alpha'}$ on $U(\sigma')$. Extending $\tilde{\alpha}$ using partition of unity to $\Omega^{k-1}(U(\sigma))$ gives the closed form (required by (a_k))

$\tilde{\omega} := d\tilde{\alpha}$ since $\tilde{\omega} = d\tilde{\alpha}|_{\partial\sigma} = \omega$ by construction of α' and $\tilde{\alpha}'$.