Closed *-analytic set $X = \bigcup_{j \le r} X^{(j)}$, i.e. each $X^{(j)}$ is j-dim manifold, is analytic and Chow's Thm: in projective space X is algebraic.

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Chow's Thm: Analytic $X \subset \mathbb{CP}^n$ are algebraic.

Fact: Closed analytic $X \subset$ open $U \subset \mathbb{C}^n$ is *-analytic (due to $\mathrm{Sing}\, X$ being closed analytic of $\dim \mathrm{Sing}\, X < \dim X$).

Proof: Cone $Z \subset \mathbb{C}^{n+1}$ over X is *-analytic hence closed analytic including at $0 \in \mathbb{C}^{n+1} \Rightarrow Z = V(f_1, \ldots, f_m)$, $f_j \in \mathbb{C}\{z\}$, and each $f_j(\lambda z) = 0$ for $z \in Z$, $\forall\ \lambda \in \mathbb{C}$. Say $f_j(z) := \sum_k f_{j,k}(z)$ with $f_{j,k}(\lambda z) \equiv \lambda^k f_{j,k}(z) \Rightarrow$ each $(\frac{\partial}{\partial \lambda})^k (f_j(\lambda z))_{|\lambda=0} = f_{j,k}(z) = 0$ on Z , i.e. Z is a zero set of finitely many $f_{j,k}(z)$ (via Hilb. Thm). ■

Plan of Proof for Main Thm: *-analytic $\Rightarrow$ analytic

Induction on dimension r of X. $X = X^{(r)} \cup X'$ with $\dim X' < r$. We may assume X' is analytic and $\overline{X^{(r)}} \subset X^{(r)} \cup X'$. We'll prove $\overline{X^{(r)}}$ is analytic. Step 1: Construct an appropriate projection map $p : \mathbb{C}^n \to \mathbb{C}^r$ which realizes a local model of $X^{(r)}$. Step 2: Use this projection map to construct analytic functions whose common zeroes are exactly points in $\overline{X^{(r)}}$ and thus complete the proof.

To construct this projection p, we first need two results.

In what follows, $X_0 = X^{(r)}$ and $X_1 = X'$.

Proper projections $p_{|X} : X \to$ open $V \subset \mathbb{C}^n$:

Proposition: For analytic $X \subset U \subset \mathbb{C}^{n+r}$, set U open with

$p(U) \subset V$, set $p(X)$ is analytic, $p_{|X} : X \to V$ finite-to-one.

Proof: Say $r = 1$ ($\Rightarrow$ general case). Let $y \in V \Rightarrow X \cap p^{-1}(y) =$

$\{a_1, \ldots, a_k\}$ being compact. Let disjoint open $U_i \ni a_i \Rightarrow \ \exists$ open

$y \in V_1 \subset V$ s.th. $p^{-1}(V_1) \subset \bigcup_i U_i$; $X \cap p^{-1}(V_1) \cap U_i =: Z_i$.

Say $y = 0$, $a_i = 0 \in Z_i = V(f_0, \ldots, f_m)$, $f_i \in \mathbb{C}\{z\}[w]$, f_0 monic

deg $= d >$ deg f_i for $i > 0$. Res $(f_0, \sum_{i=1}^m t_i f_i) = \sum_{|\alpha|=d} t^\alpha R_\alpha$

$=:$ R(z, t) , R_α converging on an open $0 \in V_2 \subset V_1 \Rightarrow$ near 0 holds $a \in p(Z_i)$ iff $\mathrm{R}(a,t) = 0 \ \forall \ t \in \mathbb{C}^m$ iff all $R_\alpha(a) = 0$. ∎

Lemma: $X_1 \ni 0$ closed analytic in open ball $U \subset \mathbb{C}^n$ around 0.

X_0 closed analytic in $U \backslash X_1$. Either $X_0 \cup X_1 = U$ or exists a line

l through 0 and ball $U_1 \subset U$ around 0, $X_1 \cap l \cap U_1 = \{0\}$ and

$X_0 \cap l \cap U_1$ is countable with the only limit point being $0 \in \mathbb{C}^n$.

Proof: Take $P \in U \backslash (X_0 \cup X_1)$ if it is nonempty. Let l be the line

joining P and 0. Then $X_1 \cap l$ is analytic and countably discrete in

$U \cap l$ and $X_0 \cap l$ analytic and countable in $U \backslash X_1 \cap l$ with limit

points in $X_1 \cap l$. Near 0, take U_1 such that $U_1 \cap X_1 \cap l = \{0\}$. ∎

$p_{|X_0}$ off $p^{-1}p(X_1) \cup \mathrm{Cr}p_{|X_0}$ is a finite covering

Step 1: Assuming $X_0 \cup X_1 \subsetneqq U$, by Lemma $\exists l$, $\exists$ projection $p : \mathbb{C}^n \to \mathbb{C}^{n-1}$ s.th. $l = p^{-1}(0)$. $(X_0 \cup X_1) \cap l \cap U_1$ compact $\Rightarrow$ $\exists$ nbhds $U_2 \subset U_1$ and $V \subset \mathbb{C}^{n-1}$ of 0 s.th. $p(U_2) \subset V$ and res $p : (X_0 \cup X_1) \cap U_2 \to V$ is proper. By Proposition, we have: $Y_1 := p(X_1 \cap U_2) \subset V$ is analytic and res $p : X_1 \cap U_2 \to Y_1$ is finite-to-one; $Y_0 := p(X_0 \cap U_2 - p^{-1}(Y_1)) \subset V - Y_1$ is analytic and res $p : (X_0 \cap U_2 - p^{-1}(Y_1)) \to Y_0$ is finite-to-one; finally, all fibres $p^{-1}(y)$ are countable.

Keep projecting in this way until the image contains a nbhd of 0.

So $\exists\ p : \mathbb{C}^n \to \mathbb{C}^r$ and open nbhds of 0: $U_1 \subset U$, $V \subset \mathbb{C}^r$, and

$p(U_1) \subset V$ s.th. res $p : (X_0 \cup X_1) \cap U_1 \to V$ is proper and onto

with countable fibres; $Y_1 := p(X_1 \cap U_1) \subset V$ is analytic and both

res $p : X_1 \cap U_1 \to Y_1$ and res $p : X_0 \cap U_1 - p^{-1}(Y_1) \to V - Y_1$

are finite-to-one. Now apply this to $X_0 = X^{(r)}$ and $X_1 = X'$.

$V - Y_1 \subset V$ is open dense; $X^{(r)} - p^{-1}(Y_1) \subset X^{(r)}$ is open dense.

Assume V is a ball. So $V - Y_1$ is connected.

$q := p_{|_{X^{(r)}}} : X^{(r)} - p^{-1}(Y_1) \to V - Y_1$ is proper and finite-to-one.

Let $B_1 := \{J = 0\} \subset X^{(r)} - p^{-1}(Y_1)$ where J is the Jacobian of q.

$B_1 \subset X^{(r)} - p^{-1}(Y_1)$ is closed analytic. $B := q(B_1) \subset V - Y_1$ is

analytic by Proposition. By Sard's Lemma, B is a nontrivial

analytic subset of $V - Y_1$. Thus $V - Y_1 - B$ is dense in V and

$X^{(r)} - p^{-1}(B \cup Y_1)$ is dense in $X^{(r)}$. Note $V - Y_1 - B$ is still

connected. Let $\pi := \operatorname{res} q : X^{(r)} - p^{-1}(B \cup Y_1) \to V - Y_1 - B$.

So locally on the source π is an iso. with constant size of fibre,

i.e., a finite unramified covering.

Beautiful construction from linear algebra

Lemma. Let $p : \mathbb{C}^n \to \mathbb{C}^r$ be any linear projection. For any $\{x_j\}_{j=0}^{j=d}$ in $\mathbb{C}^n$ where $x_0 \neq x_j$ and $p(x_0) = p(x_j)$ for all $1 \leq j \leq d$, exists $l \in (\mathbb{C}^n)^*$ s.th. $l(x_0) \neq l(x_j)$ for all $1 \leq j \leq d$.

Proof. If $\{l_\alpha\}$ are any $(n-r-1)d+1$ linear functionals in general position w.r. to the $(n-r)$-dimensional subspace $p^{-1}(0)$, then $\exists\ \alpha$ s.th. l_α has the desired property. If not, for all α, exists $j(\alpha)$ s.th. $l_\alpha(x_0) = l_\alpha(x_{j(\alpha)})$. So exists j_0 s.th. $l_\alpha(x_0) = l_\alpha(x_{j_0})$ for $n-r$ α's. By the linear independence of these l_α, $x_0 = x_{j_0}$?! ■

Constructing analytic equations defining $\overline{X^{(r)}}$.

Step 2: Let d be the number of sheets in the covering π. Choose a linear functional l on $\mathbb{C}^n$. $\forall\ 1 \leq j \leq d$, $\forall\ y \in V - Y_1 - B$, let $a_j(y)$ be the j-th elem. sym. poly. of $l(x_1), \cdots, l(x_d)$ with $\{x_1, \cdots, x_d\} = \pi^{-1}(y)$. Then all a_j are analytic on $V - Y_1 - B$. For every compact $K \subset V$, since $p_{|X}$ is proper, $X \cap p^{-1}K$ is compact. So $l(x)$ is bounded on $X^{(r)} \cap p^{-1}K$. Thus all a_j are bounded on $K \cap (V - Y_1 - B)$. By Riemann Extension Thm, all a_j extend to analytic functions on V.

Let $F_l(x) := l(x)^d + \sum_{1 \leq j \leq d} (-1)^j a_j(p(x)) \cdot l(x)^{d-j}$. Then F_l is analytic on $p^{-1}(V)$. $F_l \equiv 0$ on $X^{(r)} - p^{-1}(Y_1 \cup B)$. Hence $F_l \equiv 0$ on $\overline{X^{(r)}}$. Let $x \in p^{-1}(V) - \overline{X^{(r)}}$ and let $y = p(x)$. Let $y = \lim y_k$, $y_k \in V - Y_1 - B$. Thus $\pi^{-1}(y_k) = \{x_k^{(1)}, \cdots, x_k^{(d)}\}$ and because res $p : \overline{X^{(r)}} \to V$ is proper, we can pass to a subseq s.t. for all $j = 1, \cdots, d$, $x_k^{(j)}$ has a limit $x^{(j)}$ as $k \to \infty$. Thus $x^{(j)} \in \overline{X^{(r)}}$ so $x \neq x^{(j)}$ for any j. By Lemma, $\exists$ l s.t. $\forall$ j, $l(x) \neq l(x^{(j)})$. Fix l.

$I(x_k^{(1)}), \cdots, I(x_k^{(d)})$ are the complete set of roots of the poly

$t^d + \sum_{1 \le j \le d} (-1)^j a_j(y_k) t^{d-j}$. So $I(x^{(1)}), \cdots, I(x^{(d)})$ are the only

roots of the polynomial $t^d + \sum_{1 \le j \le d} (-1)^j a_j(y) t^{d-j}$. Hence,

$F_I(x) \neq 0$. $\blacksquare$

Corollary. $X \subset \mathbb{P}^n$ be an r-dim proj variety and let $Y \subset X$ be closed alg proper. Then $X - Y$ is connected in classical top.

Proof. If $X - Y = Z_1 \cup Z_2$, Z_i open and closed in $X - Y$, then let $S = \mathrm{Sing}\, X$. Can stratify $Z_1 \cup Y \cup S$ by taking $X^{(r)} = Z_1 \backslash (S \cup Y)$ and $X^{(i)}$ some suitable stratification of $S \cup Y$ for $0 \leq i \leq r - 1$.

Then $Z_1 \cup S \cup Y$ and $Z_2 \cup S \cup Y$ are algebraic by Chow's theorem.

$X = (Z_1 \cup S \cup Y) \cup (Z_2 \cup S \cup Y)$ and so X is not irreducible. ?! $\blacksquare$

Stronger connectedness result on transverse spaces

Proposition: $X \subset \mathbb{P}^n$ projective variety with dimension r.

$M^{n-r-1} \subset \mathbb{P}^n$ linear space disjoint from X. $p : X \to \mathbb{P}^r$ projection

from M, Line $l \subset \mathbb{P}^r$ meets $B := \{x \in \mathbb{P}^r \mid p \text{ not smooth over } x\}$

transversely, s.th. $X \backslash p^{-1}(B) \to \mathbb{P}^r \backslash B$ finite-sheeted connected

covering space. $\Rightarrow p^{-1}(l \backslash B) \to l \backslash B$ is a connected covering space.

Proof: Pick $x_0 \in l \backslash B$, and p_{x_0} the projection $\mathbb{P}^r \backslash \{x_0\} \to \mathbb{P}^{r-1}$.

Take B_0 set of non-smooth points of $p_{x_0}|_B$.

Then $B\backslash p_{x_0}^{-1}(B_0) \to \mathbb{P}^{r-1}\backslash B_0$ is a finite-sheeted covering space.

Take $l_y := p_{x_0}^{-1}(y) \cup \{x\}$ for $y \in \mathbb{P}^{r-1}$. Then $\mathbb{P}^{r-1}\backslash B_0$ is the lines

l_y meeting B transversely, $l = l_{y_0}$ for some y_0. $l_y \cap B$ is finite and

continuously varies over $\mathbb{P}^{r-1}\backslash B_0 \Rightarrow l_y\backslash B$ are diffeomorphic.

Suppose $p^{-1}(l\backslash B)$ disconnected, then $p^{-1}(l_y\backslash B)$ is disconnected.

Take $A_{y,z}$ a connected component of $p^{-1}(l_y\backslash B)$ containing some

z, then $\bigcup_y A_{y,z}\backslash p^{-1}(x)$ is clopen in $X\backslash p^{-1}(B \cup \{x_0\} \cup p_{x_0}^{-1}(B_0))$.

This contradicts the previous corollary. $\blacksquare$

Corollary: $X^r \subset \mathbb{P}^n$, exists a linear subspace L, $\dim L = n - r + 1$

such that $X \cap L$ is an irreducible curve and meet transversely.

Proof: Use the notation from Proposition, take $p_M : \mathbb{P}^n - M \to \mathbb{P}^r$

projection with center M. Then $L := p_M^{-1}(l) \cup M$ has dimension

$n - r + 1$ and $L \cap X = p^{-1}(l)$ is irreducible. Take $z \in p^{-1}(l \backslash B)$,

z is smooth and we have $\dim(T_{z,X} + T_{z,L}) = n$. ■