Hilbert Polynomials and Dimension Theory for Abelian Rings

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Graded rings $A = \bigoplus_{n=0}^{\infty} A_n$ and A-modules M :

Def. : A graded if A_n are abelian groups and $A_n \cdot A_m \subseteq A_{n+m}$, so A_0 is subring. $M = \bigoplus_{n=0}^{\infty} M_n$ is graded if $A_m \cdot M_n \subseteq M_{m+n}$.

Lemma 1 : A noetherian iff A_0 is and A is A_0-finitely generated.

Proof : $\Leftarrow$ is via Hilbert's Basis Thm. To show $\Rightarrow$ pick $x_j \in A_{m_j}$, $1 \leq j \leq s$, that generate ideal $A_+ := \bigoplus_{i>0} A_i$ over A and show $A_n \subseteq A_0[x_1, ..., x_s]$ by induction on n : true for $n = 0$ and say for $i < n$. If $x \in A_n$ then $x = \sum_{i=1}^{s} a_i x_i$ with $a_i \in A_{n-m_i}$ or $= 0$ and since $n - m_i < n \Rightarrow a_i \in A_0[x_1, ..., x_s] \Rightarrow x \in A_0[x_1, ..., x_s]$. ■

Part I: Hilbert series $P(M, t)$ for noetherian A .

Def. : λ is additive if $\lambda(N) = \lambda(M) + \lambda(L)$ when $L = N/M$.

Below all graded A-modules are A-finite i.e. finitely generated.

Thm. 2: Let $M = \bigoplus M_n$ be $A_0[x_1, ..., x_s]$-module, $x_i \in A_{k_i}$. Then

$P(M, t) := \sum_{n\geq 0} \lambda(M_n)t^n = f(t)/\prod_{i=1}^{s}(1 - t^{k_i})$ for $f \in \mathbb{Z}[t]$.

Proof : Induction on s . For $s = 0$, M A_0-finite $\Rightarrow$

$M_n = 0$ for $n \gg 0$. Say true for $s - 1$. Let $x_s : M_n \to M_{n+k_s}$ be

'times x_s' homomorphism, $K_n := \ker x_s$, $L_{n+k_s} :=$ coker x_s , i.e.

$0 \to K_n \to M_n \xrightarrow{x_s} M_{n+k_s} \to L_{n+k_s} \to 0$ is exact $\forall$ n .Then

$\lambda(K_n) - \lambda(M_n) + \lambda(M_{n+k_s}) - \lambda(L_{n+k_s}) = 0 \quad (*)$. So $L := \bigoplus L_n$, $K := \bigoplus K_n$ are graded A/x_sA-modules as x_s annihilates each. M is A-finite module $\Rightarrow$ K and L are A/x_sA-finite and $A/x_sA = A_0[\overline{x_1}, ..., \overline{x_{s-1}}]$. Summation over n of $(*)$ times t^{n+k_s} gives gives $(1 - t^{k_s})P(M, t) = P(L, t) - t^{k_s} P(K, t) + g(t)$, for a $g(t)$ in $\mathbb{Z}[t]$. By induction: $P(L, t) = f_L(t) / \prod_{i=1}^{s-1}(1 - t^{k_i})$, $P(K, t) = f_K(t) / \prod_{i=1}^{s-1}(1 - t^{k_i})$ with f_L and $f_K \in \mathbb{Z}[t]$. Therefore $P(M, t) = [f_L - t^{k_s} f_K + g(t) \prod_{i=1}^{s-1}(1 - t^{k_i})] / \prod_{i=1}^{s}(1 - t^{k_i})$. ■

Hilbert polynomial $g(n)$

Corollary 3 : Let $k_i = 1 \; \forall \; i$, $b_n := \lambda(M_n)$, and $f(1) \neq 0$. Then $\exists \; g \in \mathbb{Q}[t]$ s.th. $g(n) = b_n$ for $n \gg 0$ and $\deg g = s - 1 =: d - 1$.

Proof : Let $f(t) = \sum_{k=0}^{N} a_k t^k$ with $a_k \in \mathbb{Z}$. As $(1-t)^{-d} = \sum_{k \geq 0} \binom{d+k-1}{d-1} t^k$, then $b_n = \sum_{k=0}^{n} a_k \binom{d+n-k-1}{d-1}$. Take $g(n) := \sum_{k=0}^{N} a_k \binom{d+n-k-1}{d-1}$. Then $g(n) = b_n \; \forall \; n \geq N$, and leading coefficient is $\sum a_k/(d-1)! \neq 0$ so $\deg g = d - 1$. ■

Def. : g is the Hilbert polynomial of M with respect to λ . We will use this for $\lambda(M) = l(M) :=$ length of a composition series, which we will define now.

Definitions ; below $\mathfrak{a}$ an ideal of A

Def. : $\mathfrak{a}$ is a primary ideal if $xy \in \mathfrak{a}$ and $x \notin \mathfrak{a}$, then $y \in \sqrt{\mathfrak{a}}$.

Example : Every prime ideal is primary.

Fact : If $\mathfrak{a}$ is primary, then $\sqrt{\mathfrak{a}}$ is prime (easy exercise).

Def. : If $\mathfrak{a}$ is primary, then $\mathfrak{a}$ is $\mathfrak{p}$-primary for $\sqrt{\mathfrak{a}} = \mathfrak{p}$.

Def. : A module M is artin if $M_0 \supseteq M_1 \supseteq \ldots$ is a descending chain of submodules, then it stabilizes. Equivalently, every nonempty set of submodules has a minimal element, e.g. $A = \mathbb{C}$, $\dim_A M < \infty$

Def. : A ring B is artin if it is artin as a B-module.

Examples : i) A finite (as a set) $\mathbb{Z}$-module is artin, but $\mathbb{Z}$ is not

ii) If k is a field, then $k[t]/(t^n)$ is artin for all $n > 1$.

iii) $\mathbb{Z}[1/p]/\mathbb{Z}$ (p prime) is not noetherian, but artin $\mathbb{Z}$-module.

Fact : If M is artin, then its sub- and quotient modules are artin.

Fact : If B is artin and M is B-finite, then M is artin.

Def. : A composition series is a chain $M = M_0 \supsetneq M_1 \supsetneq \ldots \supsetneq$

$M_n = (0)$ s. th. $\forall\ i$, the only proper submodule of M_i/M_{i+1} is 0 .

Example : Let V be a vector space with basis $\{x_1, ..., x_k\}$. Then

$\{M_{k-n} := \text{span } (x_1, ..., x_{k-n})\}_{n=0}^{k}$ is a composition series for V .

Fact (A&M, 6.7): Any two composition series have same length.

Fact (A&M 6.8): M has a composition series iff M is artin and noetherian (it is an easy exercise).

Def. : Let $l(M)$ denote the length of a composition series of M .

Example : In the previous example, length would be dimension.

Fact (A&M, 6.9): Length of a module is an additive function.

Def. : A sequence of submodules $\{M_n\}$ of M is an $\mathfrak{a}$-filtration on M if $M = M_0 \supseteq M_1 \supseteq ...$ and $\mathfrak{a}M_i \subseteq M_{i+1}$. Filtration is called stable if $\mathfrak{a}M_i = M_{i+1}$ for $i \gg 0$.

Example : $M_n := \mathfrak{a}^n M$ is a stable $\mathfrak{a}$-filtration on M .

Def. : Krull dim $A := \sup\{n : \exists\ \mathfrak{p}_0 \subsetneq ... \subsetneq \mathfrak{p}_n,\ \mathfrak{p}_i$ prime$\}$.

Fact : If k is a field and domain A is a finitely generated k-algebra, then $\dim A = tr.d._k$ of the fraction field of A.

Def. : $x \in A$ is regular if $xy = 0$ for some $y \in A$, then $y = 0$.

Def. : Let $\mathfrak{p}$ be a prime ideal. The height of $\mathfrak{p}$ is $ht(\mathfrak{p}) :=$

$\sup\{r : \exists\ \mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_r = \mathfrak{p},\ \mathfrak{p}_i$ prime$\}$ and height of any ideal $\mathfrak{a}$ is

$ht(\mathfrak{a}) := \min\{ht(\mathfrak{q}) : \mathfrak{a} \subseteq \mathfrak{q}$ prime$\}$.

Example : If A is local with max. ideal $\mathfrak{m}$, then $ht(\mathfrak{m}) = \dim A$.

Note : If $\mathfrak{a} \subseteq \mathfrak{b}$ are ideals, then $ht(\mathfrak{a}) \leq ht(\mathfrak{b})$.

Def. : $\mathfrak{p}$ is a minimal prime if it is among all primes and $\mathfrak{p}$

is a minimal prime for an ideal $\mathfrak{a}$ if it is minimal among all primes

containing $\mathfrak{a}$.

Stable $\mathfrak{a}$-filtrations $\{M_n\}$ on M :

Lemma 4: If $\{M'_n\}$ another stable $\mathfrak{a}$-filtration, then $\exists\ n_0 \geq 0$ s. th. $M_{n+n_0} \subseteq M'_n$ and $M'_{n+n_0} \subseteq M_n\ \forall\ n \geq 0$.

Proof : Wlog, $M'_n := \mathfrak{a}^n M$. By induction on n , it is easy to see that $\mathfrak{a}^n M \subseteq \mathfrak{a} M_{n-1} \subseteq M_n$. As $\{M_n\}$ stable, $\exists\ n_0$ s. th. $\mathfrak{a} M_n = M_{n+1}\ \forall\ n \geq n_0 \Rightarrow M_{n+n_0} = \mathfrak{a}^n M_{n_0} \subseteq \mathfrak{a}^n M$. $\blacksquare$

Artin-Rees Lemma : Let $M' \subseteq M$ be a submodule. Then $(M' \cap M_n)$ is a stable $\mathfrak{a}$-filtration on M' .

Proof : $(M' \cap M_n)$ is an $\mathfrak{a}$-filtration: $\mathfrak{a}(M' \cap M_n) \subseteq \mathfrak{a}M' \cap \mathfrak{a}M_n \subseteq M' \cap M_{n+1}$. Let $N_n := M' \cap M_n$, $A^* := \bigoplus_{n \geq 0} \mathfrak{a}^n$, $M^* := \bigoplus_{n \geq 0} M_n$, $N^* := \bigoplus_{n \geq 0} N_n \subseteq M^*$, and $\mathfrak{a} = (x_1, ..., x_r)$. Then $A^* = A[x_1, ..., x_r]$ is noetherian. $\{M_n\}$ stable $\Rightarrow$ M^* is A^*-finite so N^* is A^*-finite, say generated by $\bigoplus_{j=0}^{k} N_j$. For $n \geq k$, $m \in N_n$ and n_{ij} generators in N_j, $j \leq k$, $\Rightarrow m = \sum a_{ij} n_{ij}$ with $a_{ij} \in \mathfrak{a}^{n-j}$. Thus $m \in \mathfrak{a}^{n-k} N_k$ as $\mathfrak{a}^{n-j} \subseteq \mathfrak{a}^{n-k}$. ■

Part II: Applications for local noetherian A .

Below $\mathfrak{m}$ is the maximal ideal of A , $\{M_n\}$ stable $\mathfrak{q}$-filtration, $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal, $G(A) := \bigoplus \mathfrak{q}^n/\mathfrak{q}^{n+1}$ and $G(M) := \bigoplus M_n/M_{n+1}$.

Prop. 5 : i) $g(n) := l(M/M_n) < \infty \; \forall \; n$;

ii) $g \in \mathbb{Q}[n]$ for $n \gg 0$ of deg. $\leq s :=$ least # of generators of $\mathfrak{q}$;

iii) $\deg g$ and its leading coeff. depend only on M and $\mathfrak{q}$.

Proof : i) As M_{n-1}/M_n is A-finite and annihilated by $\mathfrak{q}$, it is $A/\mathfrak{q}$-finite. As $A/\mathfrak{q}$ is noetherian and artin, M_{n-1}/M_n has finite length so $g(n) := l(M/M_n) = \sum_{r=1}^{n} l(M_{r-1}/M_r) < \infty$.

ii) Let $\mathfrak{q} = (x_1, ..., x_s)$ and $\overline{x_i}$ image of x_i in $\mathfrak{q}/\mathfrak{q}^2$. Then $G(A) = (A/\mathfrak{q})[\overline{x_1}, ..., \overline{x_s}]$ so $f(n) := l(M_n/M_{n+1}) \in \mathbb{Q}[n]$ of deg. $\leq s-1$ for $n \gg 0$ (Cor. 3). Fix k large.

Fact : $\sum_{i=0}^{n} i^m$ is a polynomial in n of deg. $\leq m+1$ (Faulhaber).

We have $g(n) - g(k) = \sum_{i=k}^{n-1}(g(i+1) - g(i)) = \sum_{i=k}^{n-1} f(i) = \sum_{i=k}^{n-1} \sum_{m=0}^{s-1} a_m i^m = \sum_{m=0}^{s-1} a_m \sum_{i=k}^{n-1} i^m \in \mathbb{Q}[n]$, of degree $\leq s$.

iii) Let $\{M'_n\}$ be a stable $\mathfrak{q}$-filtration. Then $\exists$ n_0 s.th. $M_{n+n_0} \subseteq M'_n$, $M'_{n+n_0} \subseteq M_n$ (Lem. 4). Then $g(n+n_0) \geq g'(n) := l(M/M'_n)$ and $g'(n+n_0) \geq g(n)$. Then $\lim_{n\to\infty} g(n)/g'(n) = 1$. $\blacksquare$

Dimension Theory: $d(A) = \delta(A) = \dim(A)$

Def. : $\delta(A) := \min\{s : \exists$ an $\mathfrak{m}$-primary ideal with s generators$\}$.

Lemma 6 : Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal and $g_{\mathfrak{q}}(n) := l(A/\mathfrak{q}^n)$.

Then $\deg g_{\mathfrak{q}} = \deg g_{\mathfrak{m}}$.

Proof : For some r , $\mathfrak{m}^r \subseteq \mathfrak{q} \subseteq \mathfrak{m}$ so $\mathfrak{m}^{rn} \subseteq \mathfrak{q}^n \subseteq \mathfrak{m}^n \ \forall \ n$. Then

$g_{\mathfrak{m}}(n) \leq g_{\mathfrak{q}}(n) \leq g_{\mathfrak{m}}(rn)$ for $n \gg 0$ but these are polynomials. ■

Def. : The common degree of the $g_{\mathfrak{q}}$ for an $\mathfrak{m}$-primary ideal $\mathfrak{q}$ is

denoted by $d(A)$.

Note: Prop. 5ii) $\Rightarrow \delta(A) \geq d(A)$.

Lemma 7 : If $x \in \mathfrak{m}$ is regular, then $d(M/xM) \leq d(M) - 1$.

Proof : Let $M' := M/xM$ and $N_n := xM \cap \mathfrak{q}^n M$. Then

Artin-Rees $\Rightarrow$ (N_n) is stable $\mathfrak{q}$-filtration of $xM \cong M$. We have

$0 \to xM/N_n \to M/\mathfrak{q}^n M \to M'/\mathfrak{q}^n M' \to 0$ exact $\Rightarrow$

$(g_{xM} - g_M + g_{M'})(n) = 0$. As g_{xM}, g_M have the same degree

and leading coefficient (Prop. 5iii), we have $\deg g_{M'} < \deg g_M$. ■

Prop. 8 : $d(A) \geq \dim A$.

Proof : Induction on $d(A)$. $d(A) = 0$ implies $l(A/\mathfrak{m}^n)$ const.

$\Rightarrow \mathfrak{m}^n = \mathfrak{m}^{n+1}$ for $n \gg 0$ so $\mathfrak{m}^n = 0$ (Nakayama Lemma). Then $\dim A = 0$. Assume true for $d(A) \leq d$. Let $d(A) = d + 1$, $\mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_r$ chain of primes in A and $A' := A/\mathfrak{p}_0$. Then $l(A'/\overline{\mathfrak{m}^n}) \leq l(A/\mathfrak{m}^n) \Rightarrow d(A') \leq d(A)$. Let $x \in \mathfrak{p}_1 \setminus \mathfrak{p}_0$. $0 \neq \overline{x} \in A'$ domain $\Rightarrow d(A'/\overline{x}A') \leq d(A') - 1$ (Lem. 7). Induction $\Rightarrow \dim(A'/\overline{x}A') \leq d \Rightarrow r - 1 \leq d$ as $\overline{\overline{\mathfrak{p}_1}} \subsetneq \ldots \subsetneq \overline{\overline{\mathfrak{p}_r}}$ chain of primes in A'/xA' . ■

Corollary 9 : $\dim A < \infty$. ■

Prop. 10 : $\dim(A) \geq \delta(A)$.

Proof : Let $d := \dim(A)$. It suffices to find $x_1, ..., x_d \in \mathfrak{m}$ s. th. $ht((x_1, ..., x_i)) \geq i \; \forall \; i$, since then $ht(x_1, ..., x_d) \geq d = ht(\mathfrak{m}) \Rightarrow$ $(x_1, ..., x_d)$ is $\mathfrak{m}$-primary $\Rightarrow \delta(A) \leq d$. Construct x_i inductively. Choose $x_1 \in \mathfrak{m} \setminus \cup_i \; \mathfrak{p}_{i,0}$ where $\mathfrak{p}_{i,0}$ are the minimal primes. Then $ht((x_1)) \geq 1$. Assume $x_1, ..., x_{i-1}$ are constructed. Let $\mathfrak{p}_1, ..., \mathfrak{p}_k$ be all the minimal primes of $(x_1, ..., x_{i-1})$ of height $i-1$ (if any). Choose $x_i \in \mathfrak{m} \setminus \cup_j \mathfrak{p}_j$. Let $\mathfrak{q}$ be a minimal prime of $(x_1, ..., x_i)$.

Then $\mathfrak{q}$ contains a minimal prime of $(x_1, ..., x_{i-1})$, say $\mathfrak{p}$. If $\mathfrak{p} = \mathfrak{p}_j$, for some j , then $x_i \notin \mathfrak{p} \Rightarrow ht(\mathfrak{q}) \geq i$. If $\mathfrak{p} \neq \mathfrak{p}_\mathrm{j}\ \forall\ j$ then $ht(\mathfrak{p}) \geq i$ so $ht(\mathfrak{q}) \geq i$. ■

Summary : We have just proved that all three notions of dimension are equal. In relation to our studies, if we localize $k[x_1, ..., x_n]$, then the trascendence degree of the fraction field is equal to any of the above three notions of dimension. This theory can also be extended to modules where $\dim M := \dim \text{Supp } M$.