Bezout's Theorem

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Say, $P := (P_1, \cdots, P_n) \colon \mathbb{C}^n \to \mathbb{C}^n$, assume $\#P^{-1}(0) < \infty$

Then (introductory lectures, October 2009) $\Rightarrow$

$$\infty > \dim_{\mathbb{C}} \frac{\mathbb{C}[x]}{(P) \cdot \mathbb{C}[x]} = \sum_{a \in P^{-1}(0)} \dim_{\mathbb{C}} \frac{\mathbb{C}[x]_a}{(P) \cdot \mathbb{C}[x]_a}$$

Let $\mathcal{L}P_j(x) := HP_j(0, x)$, where $HP_j(x_0, x) := x_0^{\deg P_j} \cdot P_j(\frac{x}{x_0})$.

Assume also $\{\mathcal{L}P_1(x) = \cdots = \mathcal{L}P_n(x) = 0\} = \{0\}$ $(*)$

(so called "no solutions at ∞") for $\mathbb{C}^n \hookrightarrow \mathbb{P}^n \Rightarrow$

Bezout's Theorem

$$a.\ P\colon \mathbb{C}^n \to \mathbb{C}^n \text{ is proper finite to one}$$

$$b.\ \sum_{a\in P^{-1}(0)} \deg_a^{\mathbb{C}} P = \deg P = \prod_{1\le j\le n} \deg P_j$$

Proof of (*) $\Rightarrow$ a.

Say, $\xi_k \in \mathbb{C}^n$, assume $\xi_k \nearrow \infty$ and all $|P_j(\xi_k)| \le C < \infty$

$a_k := \frac{\xi_k}{\|\xi_k\|} \in S^1 \Rightarrow \exists$ subsequence $a_k \to a \neq 0$

$\rho_k := \frac{1}{\|\xi_k\|} \to 0,\ 0 < d_j = \deg P_j$

Then it follows $\forall j, P_j(\xi_k) = [P_j(\frac{a_k}{\rho_k}) \cdot \rho_k^{d_j}] \cdot \frac{1}{\rho_k^{d_j}} = HP_j(\rho_k, a_k) \cdot \frac{1}{\rho_k^{d_j}}$

Therefore, $|HP_j(\rho_k, a_k)| \leq C \cdot \rho_k^{d_j} \Rightarrow \mathcal{L}P_j(a) = 0 \Rightarrow ?! \surd$

Now prove b. in three steps.

Step 1 is $\deg_0^{\mathbb{C}} \mathcal{L}P = \deg_0^{\mathbb{C}} HP$, where

$\mathcal{L}P := (\mathcal{L}P_1, \cdots, \mathcal{L}P_n) \colon \mathbb{C}^n \to \mathbb{C}^n$

$HP : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$

$HP(x_0, x) := (x_0, HP_1(x_0, x), \cdots, HP_n(x_0, x))$

y is a regular value for $\mathcal{L}P \Rightarrow (0, y)$ is regular value for HP

$\Rightarrow$(obviously) $\#HP^{-1}(0, y) = \#\mathcal{L}P^{-1}(y)$, which suffices.

Indeed

$$\frac{\partial HP}{\partial(x_0, x)}|_{x_0=0} = \begin{pmatrix} \underline{1}| & \underline{*} & \underline{*} & \underline{*} \\ 0| & & & \\ 0| & & \frac{\partial HP}{\partial x} & \\ 0| & & & \end{pmatrix}|_{x_0=0}$$

$$\text{and } \frac{\partial HP}{\partial x}(0, x)|_{x_0=0} = \frac{\partial \mathcal{L}P}{\partial x}(x)$$

Step 1. completed. $\surd$

Step 2 is $\deg P = \deg_0^{\mathbb{C}} HP$

suffices to find a regular value $a \in \mathbb{C}^n$ for map P and

a regular value $(b_0, b) \in \mathbb{C}^{n+1}$ for map HP

such that each $a_j = b_j / b_0^{d_j}$, where $d_j := degP_j$.

Step 2 follows since regular values are open dense sets

and map $(y_0, y_1, \cdots, y_n) \to (y_0, y_0^{d_1} \cdot y_1, \cdots, y_0^{d_n} \cdot y_n)$ is

a diffeomorphism of $\mathbb{C}^* \times \mathbb{C}^n$.

Step 2. completed. $\surd$

Step 3 is $\deg_0^{\mathbb{C}} \mathcal{L}P = \prod_j \deg P_j$. Let $\phi := \mathcal{L}P \Rightarrow \phi^{-1}(0) = \{0\}$

Proof is split into

Thm. $0 \neq \mathcal{J}_\phi = \deg_0^{\mathbb{C}} \phi \cdot \delta(x, 0) \mod (\phi) \cdot \mathbb{C}\{x\}$

(with $\delta(x, y)$ from 2 below). Claim. $\mathcal{J}_\phi = \prod_{1 \leq j \leq n} d_j \cdot \delta(x, 0)$.

Proof of Thm

1. Using Mumford's Lemma, $\exists$ open $U \ni 0$, $V \ni 0$ s.th.

$\phi := \phi|_U \colon U \to V$ proper. (and $U \cap \phi^{-1}(0) = \{0\}$)

Using Key Lemma from Mitsuru's talk

$\Rightarrow Z := \mathrm{Cr.Val}(\phi) := \phi(\{\mathcal{J}_\phi = 0\}) \subset V$ closed analytic.

2. $A(x,y) := \int_0^1 \frac{\partial \phi}{\partial x}(tx + (1-t)y)dt \Rightarrow A(x,y) \cdot (x-y) =$

$\phi(x) - \phi(y)$ and $\mathcal{J}_\phi(x) = \delta(x,x)$, where $\delta(x,y) := \det A(x,y)$

3. Let $h(x,z) := \sum_{\{y:\phi(y)=z\}} \delta(x,y) \Rightarrow h$ is $\mathbb{C}-$analytic on $V \setminus Z$

and bounded $\Rightarrow$ h analytic on V (Riemann Extension Thm)

4. $h(x,0) = \deg_0^{\mathbb{C}} \phi \cdot \delta(x,0)$ due to $\phi|_U^{-1}(0) = \{0\}$.

5. $\mathcal{J}_\phi(x) - h(x,z) = 0$ if $z = \phi(x) \notin Z$

$\Rightarrow \mathcal{J}_\phi(x) - h(x,z) \in (z - \phi(x)) \cdot \mathbb{C}\{x,z\}$

6. Set $z = 0 \Rightarrow \mathcal{J}_\phi(x) - (\deg_0^{\mathbb{C}} \phi) \cdot \delta(x,0) \in (\phi) \cdot \mathbb{C}\{x\}$, as required.

Recall.

$\deg \mathcal{L}P_j = \deg P_j = d_j,\ \phi_j := \mathcal{L}P_j \Rightarrow \frac{\partial \phi_j}{\partial x_i}(tx) = t^{d_j-1}\frac{\partial \phi_j}{\partial x_i}(x)$

Claim $\mathcal{J}_\phi = d_1 \cdots d_n \cdot \delta(x, 0)$

Cor. Since $\mathcal{J}_\phi \notin (\phi) \cdot \mathbb{C}\{x\}$ (Slides 9-11) $\Rightarrow \deg_0^{\mathbb{C}} \phi = d_1 \cdots d_n$

Proof. Explicit calculation

$$A(x,0) := \int_0^1 \{\frac{\partial \phi_j}{\partial x_i}(tx)\}dt = \int_0^1 \begin{bmatrix} t^{d_1-1} & & 0 \\ & \ddots & \\ 0 & & t^{d_n-1} \end{bmatrix} (\frac{\partial \phi}{\partial x}(x))dt$$

$$= \begin{bmatrix} \frac{1}{d_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{d_n} \end{bmatrix} (\frac{\partial \phi}{\partial x}(x)) \qquad \surd$$

Thm. $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0), \phi^{-1}(0) = \{0\} \Rightarrow \mathcal{J}_\phi(x) \notin (\phi) \cdot \mathbb{C}\{x\}$

Thm $\Leftarrow$ Claim. $h \in \mathbb{C}\{x\} \Rightarrow H_h(z) := \sum_{x \in \phi^{-1}(z)} \frac{h(x)}{\mathcal{J}_\phi(x)} \in \mathbb{C}\{z\}$

Indeed, $H_{\mathcal{J}_\phi} \equiv \deg_0^{\mathbb{C}} \phi \neq 0$, while $g = \sum_j \phi_j \cdot h_j \in (\phi) \cdot \mathbb{C}\{x\} \Rightarrow$

$H_g(z) = \sum_j z_j \cdot H_{h_j}(z) \in (z) \cdot \mathbb{C}\{z\}$, as required.

Proof of Claim $X := \phi|_U(\{\mathcal{J}_{\phi|U} = 0\})$.

(Note $X = \{\prod_{x \in \phi|_U^{-1}(z)} \mathcal{J}_\phi(x) = 0\} \ \subset V$)

$\phi|_{\mathcal{J}_\phi = 0} : \{\mathcal{J}_\phi = 0\} \to X$ proper fin. to one $\Rightarrow \dim_{\mathbb{C}} Z < \dim_{\mathbb{C}} X$

where $Z = \phi|_U(Sing\{\mathcal{J}_\phi = 0\}) \cup Sing X \cup Cr.Val(\phi|_{\mathcal{J}_\phi = 0})$ analytic.

<u>Step 1.</u> $\forall a \in \{\mathcal{J}_{\phi|_U} = 0\} \setminus \phi|_U^{-1}(Z)$, $\exists$ local coord. changes s.th.

$a = 0, \phi(a) = 0, \{\mathcal{J}_\phi = 0\} = \{x_1 = 0\}, X = \{z_1 = 0\}$

$\Rightarrow \phi(0, x_2, \cdots, x_n) = (0, x_2, \cdots, x_n)$. Also $\mathcal{J}_\phi \approx x_1^k$

Say $\phi_1(x_1, x_2, \cdots, x_n) := x_1^d \cdot g(x), g(x) \notin (x_1) \cdot \mathbb{C}\{x\} \quad (d \geq 1)$

and $\theta(x) := \det[\frac{\partial(\phi_2, \cdots, \phi_n)}{\partial(x_2, \cdots, x_n)}]_{|x_1=0} = 1 \Rightarrow k = d - 1, g(0) \neq 0$.

Say, $h(x)^d = g(x) \Rightarrow$ (coord. change $x \mapsto (x_1 h(x), \phi_2, \cdots, \phi_n)$)

Near a, we may assume $\phi(x) = (x_1^d, x_2, \cdots, x_n)$

<u>Step 2.</u> $h \in Hol(U) \Rightarrow H_h \in Hol(V - X)$.

But moreover, $\sum_{x_1 : x_1^d = z_1} x_1^{k-d+1} = 0 \ \forall k + 1 \notin d \cdot \mathbb{Z}_+$

with $h(x)=\sum_{k=0}^{\infty} h_k(x_2,\cdots,x_n)x_1^k$ near a $(\mathcal{J}_\phi(x)=x_1^{d-1}\cdot d)\Rightarrow$

Near a, $\sum_{x\in\phi^{-1}(z)} \frac{h(x)}{\mathcal{J}_\phi(x)} = \sum_{k=0}^{\infty} h_k(z_2,\cdots,z_n)\sum_{x_1^d=z_1} \frac{x_1^{k-d+1}}{d} =$

$= \sum_{(k-d+1)|d} h_k(z_2,\cdots,z_n)\cdot z_1^{\frac{k-d+1}{d}}$

$\Rightarrow H_h(z)$ extends as holom. to $V\setminus Z$, $\dim_{\mathbb{C}} Z \le n-2$

(Hartog's Thm) $H_h\in Hol(V)$, done. $\surd$