Differential forms and the de Rham cohomology - Part I

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October 30, 2009

I. Review—Triangulation of Manifolds

$M =$ smooth, compact, oriented n-manifold. Can *triangulate* M, i.e. $\exists$ simplicial n-complex K, homeomorphism $\pi : K \to M$ s.t. $\forall \sigma \in K$ $\exists$ coordinate neighbourhood (U, f) near $\overline{\sigma}$ s.t. $f^{-1} \circ \pi$ is affine in σ. Write σ instead of $\pi(\sigma)$. $\sigma^k \in K$ orientable by ordering vertices: $\overline{\sigma^k} = q_0 \cdots q_k$.

Given $\{q_{\lambda_0}, \ldots, q_{\lambda_r}\} \subseteq \{q_0, \ldots, q_k\}$, $\tau = q_{\lambda_0} \cdots q_{\lambda_r}$ called *r-face* of σ. Inductively, orientation of $\tau_i = q_0 \cdots q_{i-1} q_{i+1} \cdots q_k$ agrees with orientation of σ iff i is even. Write $F(\sigma) =$ set of faces of σ, boundary $\partial\sigma = \overline{\sigma} \backslash \sigma$ with appropriate orientation.

k-chain $A = \sum a_i \sigma_i^k \in \Sigma_k$ is a formal sum of k-simplices. $\partial A = \sum a_i \partial \sigma_i^k$. For every simplex $\sigma \in \Sigma_k$, there is a "cosimplex" $\sigma \in \Sigma^k$ defined by $\sigma \cdot \sigma = 1$, $\sigma \cdot \tau = 0$ for $\tau \neq \sigma$. Define coboundary: $(\partial^* A) \cdot B = A \cdot (\partial B)$.

Review—Differential forms

$\Lambda^k(T_pM)^* =$ set of k-linear alternating functions on $(T_pM)^k$. Differential k-form $\omega \in \Omega^k(M) : p \in M \mapsto \omega(p) \in \Lambda^k(T_pM)^*$. Let $f : M \to N$ be a smooth map, $Df =$ tangent map of f. For $\omega \in \Omega^k(N)$ define $f^*\omega \in \Omega^k(M)$:

$$(f^*\omega)(p)(v_1, \ldots, v_k) = \omega\left(f(p)\right)\left(Df(p)(v_1), \ldots, Df(p)(v_k)\right)$$

Cover compact n-manifold M with coordinate charts $U_i \subseteq \mathbb{R}^n \xrightarrow{f_i} M$, $i = 1..k$. Then $\exists$ partition of unity $\{\phi_i : M \to \mathbb{R}\}_{i=1..k}$ s.t.:

1. All ϕ_i are smooth
2. $\mathrm{Supp}\,(\phi_i) \subset U_i \forall i$, where $\mathrm{Supp}\,(\phi_i) = \mathrm{clos}\{p \in M : \phi_i(p) \neq 0\}$
3. $\sum_{i=1}^k \phi_i = 1$

For n-form $\omega = \alpha dx_1 \wedge \ldots \wedge dx_n$ in coords $x = (x_1, \ldots, x_n)$, define $\int_V \omega = \int_U \alpha dx_1 \ldots dx_n$. In general, define $\int_M \omega = \sum_{i=1}^k \int_{U_i} \phi_i \omega$

II. Stokes' Theorem

Let M be a compact oriented n-manifold with boundary, ω an $(n-1)$-form on M. Then $\int_{\partial M} \omega = \int_M d\omega$.

Proof. Let $K = \mathrm{Supp}\,(\omega)$. Only need to prove for $K \subset V = f(U)$, for coordinate chart (U, f), then use partition of unity. Write

$$\omega = \sum_{j=1}^{n} a_j dx_1 \wedge \ldots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \ldots \wedge dx_n,$$

$$d\omega = \left(\sum_{j=1}^{n} (-1)^{j-1} \frac{\partial a_j}{\partial x_j} \right) dx_1 \wedge \ldots \wedge dx_n.$$

Two cases to consider: $f(U) \cap \partial M = \varnothing$ and otherwise.

Stokes' Theorem—$f(U) \cap \partial M = \varnothing$

(1) $f(U) \cap \partial M = \varnothing$. Then $\int_{\partial M} \omega = 0$, so we must show

$$\int_M d\omega = \int_U \left(\sum (-1)^{j-1} \frac{\partial a_j}{\partial x_j} \right) dx_1 \ldots dx_n = 0.$$

WLOG, $f^{-1}(K) \subset \operatorname{int}(Q)$, where $Q = \{x \in H^n : x_j^1 \leq x_j \leq x_j^0\}$ ($H^n = \{x \in \mathbb{R}^n : x_0 \leq 0\}$), for x_j^1, x_j^0 such that $f^{-1}(K) \subset \operatorname{int}(Q)$. Then,

$$\int_U \left(\sum (-1)^{j-1} \frac{\partial a_j}{\partial x_j} \right) dx_1 \ldots dx_n = \sum (-1)^{j-1} \int_Q \frac{\partial a_j}{\partial x_j} dx_1 \ldots dx_n$$
$$= \sum (-1)^{j-1} \int_Q [a_j(x_1, \ldots, x_j^0, \ldots, x_n)$$
$$- a_j(x_1, \ldots, x_j^1, \ldots, x_n)] dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_n = 0$$

Stokes' Theorem—$f(U) \cap \partial M \neq \varnothing$

(2) $f(U) \cap \partial M \neq \varnothing$. Then

$$\omega|_{\partial M} = a_1(0, x_2, \ldots, x_n) dx_2 \wedge \ldots \wedge dx_n$$

Set Q as above, with $x_1^0 = 0$. Then,

$$\begin{aligned}
\int_M d\omega &= \sum (-1)^{j-1} \int_Q \frac{\partial a_j}{\partial x_j} dx_1 \cdots dx_n \\
&= \int_Q \left[a_1(0, \ldots, x_n) - a_1(x_1^1, \ldots, x_n) \right] dx_2 \ldots dx_n \\
&= \int_Q a_1(0, x_2, \ldots, x_n) dx_2 \cdots dx_n = \int_{\partial M} \omega
\end{aligned}$$

QED

III. Poincaré's Lemma

M is *contractible* to $p_0 \in M$ if $\exists$ smooth map $C : M \times \mathbb{R} \to M$ s.t. for all $p \in M$, $C(p, 1) = p$ and $C(p, 0) = p_0$

Poincaré's Lemma. *If M is contractible, $\omega \in \Omega^k(M)$ is closed ($d\omega = 0$) iff it is exact ($\omega = d\eta$ for some η).*

Proof. Let $\pi : M \times \mathbb{R} \to M$ be the projection map, $\overline{\omega} = C^*\omega$.

Note that $\overline{\omega}$ can be written uniquely as $\overline{\omega} = \omega_1 + dt \wedge \eta$, such that $\omega_1(v_1, \ldots, v_k) = 0$ if some $v_i \in \ker(D\pi)$, and similarly for η.

Poincaré's Lemma

Let $i_t : M \to M \times \mathbb{R}$, $i_t(p) = (p, t)$. Define $I : \Omega^k(M \times \mathbb{R}) \to \Omega^{k-1}(M)$ as follows. For $p \in M$, $v_1, \ldots, v_{k-1} \in T_pM$,

$$(I\overline{\omega})(v_1, \ldots, v_{k-1}) = \int_0^1 \left[\eta(p,t)\left(Di_t(v_1), \ldots, Di_t(v_{k-1})\right)\right] dt$$

Claim: $d(I\overline{\omega}) = \omega$

Sublemma. $i_1^*\overline{\omega} - i_0^*\overline{\omega} = d(I\overline{\omega}) + I(d\overline{\omega})$.
Note that $I(\omega_1 + \omega_2) = I(\omega_1) + I(\omega_2)$, so enough to prove for $\overline{\omega} = f dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ or $\overline{\omega} = f dt \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$.

Poincaré's Lemma—sublemma

Case 1: $\overline{\omega} = f dx_{i_1} \wedge \ldots \wedge dx_{i_k}$, we have
$d\overline{\omega} = \frac{\partial f}{\partial t} dt \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} +$ terms not containing dt.

$$
\begin{aligned}
I(d\overline{\omega})(p) &= \left(\int_0^1 \frac{\partial f}{\partial t} dt \right) dx_{i_1} \wedge \ldots \wedge dx_{i_k} \\
&= \left(f(p,1) - f(p,0) \right) dx_{i_1} \wedge \ldots \wedge dx_{i_k} \\
&= i_1^* \overline{\omega}(p) - i_0^* \overline{\omega}(p)
\end{aligned}
$$

Case 2: $\overline{\omega} = f dt \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$. Since i_t takes M to a subspace of constant t, we have $i_1^* \overline{\omega} = i_0^* \overline{\omega} = 0$, so we need $d(I\overline{\omega}) = -I(d\overline{\omega})$.

Poincaré's Lemma—sublemma

$$d\overline{\omega} = \sum_{\alpha=1}^{n} \frac{\partial f}{\partial x_\alpha} dx_\alpha \wedge dt \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_{k-1}},$$

$$(Id\overline{\omega})(p) = -\sum_{\alpha} \left(\int_0^1 \frac{\partial f}{\partial x_\alpha} dt \right) dx_\alpha \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_{k-1}}.$$

On the other hand,

$$\begin{aligned} d(I\overline{\omega})(p) =& d \left\{ \left(\int_0^1 f dt \right) dx_{i_1} \wedge \ldots \wedge dx_{i_{k-1}} \right\} \\ =& \sum_{\alpha} \left(\int_0^1 \frac{\partial f}{\partial x_\alpha} dt \right) dx_\alpha \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_{k-1}} \end{aligned}$$

Poincaré's Lemma

$$\begin{aligned} \omega = & \ (C \circ i_1)^* \omega = & \ i_1^*(C^* \omega) = & \ i_1^* \overline{\omega}, \\ 0 = & \ (C \circ i_0)^* \omega = & \ i_0^*(C^* \omega) = & \ i_0^* \overline{\omega} \end{aligned}$$

Since $d\omega = 0$, we get $d\overline{\omega} = C^* d\omega = 0$. Setting $\alpha = I\overline{\omega}$,

$$\omega = i_1^* \omega = d(I\overline{\omega}) = d\alpha.$$

QED

IV. The de Rham Chain Complex

Define $\left(Int^k(\omega)\right)(A) = \int_A \omega$.

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \Omega^{k-1}(M) & \xrightarrow{d_{k-1}} & \Omega^k(M) & \xrightarrow{d_k} & \Omega^{k+1}(M) & \longrightarrow \cdots \\
& \downarrow Int^{k-1} & & \downarrow Int^k & & \downarrow Int^{k+1} & \\
\cdots \longrightarrow & \Sigma^{k-1} & \xrightarrow{\partial^*_{k-1}} & \Sigma^k & \xrightarrow{\partial^*_k} & \Sigma^{k+1} & \longrightarrow \cdots
\end{array}$$

Diagram commutes by Stokes' Thm.

Star $\mathrm{St}(\sigma) = \bigcup_{\sigma \in F(\tau)} \tau$ open in M, $\{\mathrm{St}(p_i)\}$ cover M. Define subordinate partition of unity $\{\phi_i\}$, and define

$$\Phi^k(q_{\lambda_0} \cdots q_{\lambda_k}) = k! \sum_{i=0}^{k} (-1)^i \phi_{\lambda_i}\, d\phi_{\lambda_0} \wedge \cdots \hat{i} \cdots \wedge d\phi_{\lambda_k}$$

V. Elementary forms

Theorem.

$$\text{Supp}\,(\Phi^k\sigma) \subset \text{St}(\sigma) \tag{1a}$$

$$\Phi^k\partial^* A = d\Phi^{k-1}A \tag{1b}$$

$$\Phi^0(\sum q_i) = \mathbf{1} \tag{1c}$$

Furthermore, Φ^k is a right inverse of Int^k.
Proof. (1a) follows immediately since Supp $\phi_i \subset \text{St}(q_i)$.
For (1c), $\Phi^0(q_i) = \phi_i$, hence $\Phi^0(\sum_i q_i) = \mathbf{1}$.
Now (1b). Let $\sigma = q_{\lambda_0} \cdots q_{\lambda_k}$, then

$$d\Phi^k(q_{\lambda_0} \cdots q_{\lambda_k}) = (k+1)!\,d\phi_{\lambda_0} \wedge \ldots \wedge d\phi_{\lambda_k}.$$

Sidenote: two important identities

Recall $(\partial^*\sigma)\cdot\tau = \sigma\cdot(\partial\tau) = 1$ if $\sigma \in \partial\tau$ and 0 otherwise. Hence $\partial q_{\lambda_0}\cdots q_{\lambda_k} = \sum_s^* q_s q_{\lambda_0}\cdots q_{\lambda_k}$, where $\sum^*$ is over s s.t. $q_s q_{\lambda_0}\cdots q_{\lambda_k}$ is simplex. In particular, $\frac{1}{(k+1)!}\Phi\partial^*(q_{\lambda_0}\cdots q_{\lambda_k}) = \frac{1}{(k+1)!}\sum_s^* \Phi(q_s q_{\lambda_0}\cdots q_{\lambda_k})$.

Also note that $d\left(\sum\phi_i\right) = 0$. Since $\sum\phi_i = \sum_{i=0}^k \phi_{\lambda_i} + \sum_{s\neq\lambda_i}\phi_s$, we have

$$\sum_{i=0}^{k} d\phi_{\lambda_i} + \sum_{s\neq_{\text{any}}\,\lambda_i} d\phi_s = 0$$

We use both these identities in the following.

Elementary forms—Proof of (1b)

$$\frac{1}{(k+1)!}\sum_s^* \Phi(q_s q_{\lambda_0}\cdots q_{\lambda_k})$$

$$=\sum_s^* \left\{\phi_s d\phi_{\lambda_0}\wedge\ldots\wedge d\phi_{\lambda_k} - \sum_{i=0}^{k}(-1)^i \phi_{\lambda_i} d\phi_s \wedge d\phi_{\lambda_0}\wedge\cdots\hat{i}\cdots\wedge d\phi_{\lambda_k}\right\}$$

$$=\sum_{s\neq \text{any } \lambda_i} \phi_s d\phi_{\lambda_0}\wedge\ldots\wedge d\phi_{\lambda_k} +$$

$$\sum_{i=0}^{k}(-1)^i\phi_{\lambda_i}\sum_{j=0}^{k} d\phi_{\lambda_j}\wedge d\phi_{\lambda_0}\wedge\ldots\hat{i}\ldots\wedge d\phi_{\lambda_k}$$

$$=\sum_{s\neq \text{any } \lambda_i} \phi_k d\phi_{\lambda_0}\wedge\ldots\wedge d\phi_{\lambda_k} + \sum_{i=0}^{k}\phi_{\lambda_i} d\phi_{\lambda_0}\wedge\ldots\wedge d\phi_{\lambda_k}$$

$$=d\phi_{\lambda_0}\wedge\ldots\wedge d\phi_{\lambda_k} = \frac{1}{(k+1)!}d\Phi(q_{\lambda_0}\cdots q_{\lambda_k})$$

Elementary forms—$Int \circ \Phi^k = Id$

Now prove Φ^k is right-sided inverse of Int^k by induction. $k = 0 \Rightarrow$ consider vertex q_j and covertex q_i, so $q_i \cdot q_j = \delta_{ij}$. Note by definition $\Phi^0(q_i) = \phi_i$ and $\phi_i(q_j) = \delta_{ij}$. Also note for function f, $Int^0(f) \cdot q_j = f(q_j)$. Hence $Int^0\left(\Phi^0(q_i)\right) \cdot q_j = \delta_{ij}$, i.e. $Int^0 \circ \Phi^0 = Id$.

$k > 0 \Rightarrow$ will follow from

$$(Int^k \Phi^k \sigma) \cdot \tau = \int_\tau \Phi^k \sigma = \begin{cases} 0 & \text{if } \tau \neq \sigma \\ 1 & \text{if } \tau = \sigma \end{cases}$$

Suppose $\tau \neq \sigma$, then $\exists q_i \in F(\sigma)$ with $q_i \notin F(\tau)$, so $\phi_i \equiv 0$ in τ, proving for $\tau \neq \sigma$.

Elementary forms—$Int \circ \Phi^k = Id$

If $\sigma = \tau$, write $\partial\sigma = \psi + \ldots$. Then,

$$\int_\sigma \Phi^k \sigma \overset{!}{=} \int_\sigma \Phi^k \partial^* \psi = \int_\sigma d\Phi^{k-1}\psi = \int_{\partial\sigma} \Phi^{k-1}\psi \overset{!}{=} \int_\psi \Phi^{k-1}\psi = 1,$$

where ! uses the case $\sigma \neq \tau$ and the last equality uses inductive hypothesis.

QED

For $\sigma \in K$, $\Phi\sigma$ called an *elementary form*.

VI. Stay tuned!

Define $\mathbf{H}^k_\Sigma = \ker(\partial^*_k)/\operatorname{im}(\partial^*_{k-1})$, $\mathbf{H}^k_\Omega = \ker(d_k)/\operatorname{im}(d_{k-1})$, the differential and simplicial cohomology groups respectively. Define $\mathbf{H}^*_\Sigma = \bigoplus \mathbf{H}^k_\Sigma$, $\mathbf{H}^*_\Omega = \bigoplus \mathbf{H}^k_\Omega$. For $\mathbf{h} \in \mathbf{H}^k$, $\mathbf{h}' \in \mathbf{H}^r$, can define cup product $\mathbf{h} \smile \mathbf{h}' \in \mathbf{H}^{k+r}$ under which $\mathbf{H}^*_\Sigma$ and $\mathbf{H}^*_\Omega$ become rings.

We will show the subcomplex $(\ker \mathit{Int}^\bullet)$ is acyclic (i.e. $\ker d_k = \operatorname{im} d_{k-1}$ when restricted to $\ker \mathit{Int}^\bullet$) and use this to prove de Rham's theorem: $\mathit{Int}^\bullet$ is a ring isomorphism between $\mathbf{H}^*_\Omega$ and $\mathbf{H}^*_\Sigma$.